# Nonequilibrium Formulation of Abelian Gauge Theories

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# Abstract


This work is about a formulation of abelian gauge theories out-of-equilibrium. In contrast to thermal equilibrium, systems out-of-equilibrium are not constant in time, and the interesting questions in such systems refer to time evolution problems. After a short introduction to quantum electrodynamics (QED), the two-particle irreducible (2PI) effective action is introduced as an essential technique for the study of quantum field theories out-of-equilibrium. The equations of motion (EOMs) for the propagators of the theory are then derived from it. It follows a discussion of the physical degrees of freedom (DOFs) of the theory, in particular with respect to the photons, since in covariant formulations of gauge theories unphysical DOFs are necessarily contained.

After that the EOMs for the photon propagator are examined more closely. It turns out that they are structurally complicated, and a reformulation of the equations is presented which for the untruncated theory leads to an essential structural simplification of the EOMs. After providing the initial conditions which are necessary in order to solve the EOMs, the free photon EOMs are solved with the help of the reformulated equations. It turns out that the solutions diverge in time, i. e. they are secular. This is a manifestation of the fact that gauge theories contain unphysical DOFs. It is reasoned that these secularities exist only in the free case and are therefore "artificial". It is however emphasized that they may not be a problem in principle, but certainly are in practice, in particular for the numerical solution of the EOMs. Further, the origin of the secularities, for which there exists an illustrative explanation, is discussed in more detail.

Another characteristic feature of 2PI formulations of gauge theories is the fact that quantities calculated from approximations of the 2PI effective action, which are gauge invariant in the exact theory as well as in an approximated theory at each perturbative order, are not gauge invariant in general. A closely related phenomenon is the fact that the Ward identities, which are relations between correlation functions of different order, are not in general applicable to correlation functions which are calculated from the 2PI effective action. As an example the photon self-energy is presented, which is transverse in the exact theory as well as perturbatively at each order, but not if it is calculated starting from the 2PI effective action. It is shown that both these phenomena are caused by the complex resummation implemented by the 2PI effective action.

Finally, a concrete approximation of the 2PI effective action is presented, and the self-energies are derived from it in a form which is suitable for the practical implementation on a computer. Some results are shown which have been obtained by the numerical solution of the 2PI EOMs.

# Zusammenfassung


In dieser Arbeit geht es um eine Formulierung von abelschen Eichtheorien im Ungleichgewicht. Im Gegensatz zum thermischen Gleichgewicht sind Systeme im Ungleichgewicht in der Zeit veränderlich, und die interessanten Fragestellungen in solchen Systemen beziehen sich auf Zeitentwicklungsprobleme. Nach einer kurzen Einführung in die Quantenelektrodynamik (QED) wird als wesentliche Technik zum Studium von Quantenfeldtheorien im Ungleichgewicht die Zwei-Teilchen-irreduzible (2PI) effektive Wirkung eingeführt. Aus dieser werden die Bewegungsgleichungen für die Propagatoren der Theorie abgeleitet. Es folgt eine Diskussion der physikalischen Freiheitsgrade der Theorie, insbesondere im Hinblick auf die Photonen, denn in kovarianten Formulierungen von Eichtheorien sind notwendigerweise unphysikalische Freiheitsgrade enthalten.

Daran anschließend werden die Bewegungsgleichungen für den Photon-Propagator näher untersucht. Es stellt sich heraus, dass sie strukturell kompliziert sind, und es wird eine Umformulierung der Gleichungen präsentiert, die für die ungenäherte Theorie zu einer wesentlichen strukturellen Vereinfachung der Bewegungsgleichungen führt. Nach Angabe der zur Lösung der Bewegungsgleichungen benötigten Anfangsbedingungen werden die freien Photon-Bewegungsgleichungen mit Hilfe der umformulierten Gleichungen gelöst. Es zeigt sich, dass die Lösungen mit der Zeit divergieren, d. h. sie sind sekulär. Tatsächlich stellt sich das als Manifestation der Tatsache heraus, dass Eichtheorien unphysikalische Freiheitsgrade enthalten. Es werden Gründe dafür angegeben, dass diese Sekularitäten überhaupt nur im freien Fall existieren und damit "künstlich" sind. Es wird jedoch betont, dass, auch wenn sie kein prinzipielles, so doch sicherlich ein praktisches Problem darstellen, insbesondere für die numerische Lösung der Bewegungsgleichungen. Weiterhin wird auf den Ursprung der Sekularitäten eingegangen, für den es eine anschauliche Erklärung gibt.

Ein weiteres charakteristisches Merkmal von 2PI-Formulierungen von Eichtheorien ist die Tatsache, dass aus Näherungen der 2PI-effektiven Wirkung berechnete Größen, die sowohl in der exakten Theorie als auch in in einer genäherten Theorie perturbativ in jeder Ordnung eichinvariant sind, i. A. nicht eichinvariant sind. Ein eng verwandtes Phänomen ist die Tatsache, dass die Ward-Identitäten, die Relationen zwischen Korrelationsfunktionen unterschiedlicher Ordnung darstellen, i. A. nicht auf aus der 2PI-effektiven Wirkung berechnete Korrelationsfunktionen anwendbar sind. Als Beispiel wird die Photon-Selbstenergie angegeben, die sowohl in der exakten Theorie als auch perturbativ in jeder Ordnung transversal ist, jedoch nicht, wenn sie ausgehend von der 2PI-effektiven Wirkung berechnet wird. Es wird gezeigt, dass diese beiden Phänomene durch die komplexe Resummierung verursacht werden, die die 2PI-effektive Wirkung implementiert.

Schließlich wird eine konkrete Näherung der 2PI-effektiven Wirkung präsentiert, und es werden die Selbstenergien aus ihr in einer Form abgeleitet, die für die praktische Implementierung auf einem Computer benötigt wird. Es werden einige Resultate gezeigt, die durch die numerische Lösung der 2PI-Bewegungsgleichungen gewonnen wurden.

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# Notation and Conventions

Here we gather the notation and conventions used throughout this work unless stated otherwise.

### Units

We use natural units in which $\hbar = c = k_\mathrm{B} = 1$. It follows in particular that mass, energy, momentum and temperature all have the same dimension, $[m] = [E] = [p] = [T]$, while time has the inverse dimension of a mass, $[t] = 1/[m]$, and actions are dimensionless, $[S] = 1$.

We use the fermion mass $m^{(\mathrm{f})}$ as the only unit. We can then turn dimensionful quantities into dimensionless quantities by multiplying with powers of the mass, e. g. $\hat{p} = p/m^{(\mathrm{f})}$, $\hat{t} = m^{(\mathrm{f})}\, t$, etc.

### Vectors

We use two types of vectors: Lorentz vectors, by which we mean vectors transforming under the Lorentz group SO(3, 1) (and which we also sometimes call “four-vectors”); and spatial vectors, by which we mean vectors transforming under the rotation group SO(3). Lorentz vectors are usually denoted by small Latin letters, like $v$, while spatial vectors are denoted by small boldface letters, like $\boldsymbol{v}$. In particular, we then have $v = (v^0, \boldsymbol{v})$. We will often use the same symbol for a four-vector and for the modulus of a spatial vector, i. e. $v = |\boldsymbol{v}|$. Since we mostly use spatial vectors, there will be no risk of confusion.

### Metric

We will exclusively make use of the Minkowski metric $(g_{\mu\nu})$ in this work, which we define as

$$(g_{\mu\nu}) = \mathrm{diag}(1, -1, -1, -1) = (g^{\mu\nu})\,,$$

where $(g^{\mu\nu})$ is the inverse Minkowski metric.

The metric can be used to define a scalar product. For two four-vectors $v, w$, we define $v \cdot w = g_{\mu\nu} v^\mu w^\nu = v^0 w^0 - \boldsymbol{v} \cdot \boldsymbol{w} = v^0 w^0 - \delta_{ij} v^i w^j$.

Correspondingly, for two four-covectors $\alpha, \beta$, we employ the inverse metric to define $\alpha \cdot \beta = g^{\mu\nu} \alpha_\mu \beta_\nu$.

Since we always assume the Minkowski metric to be given, there is usually no need to discriminate between vectors and covectors since the metric can always be used to map one to the other.

Further note that we have

$$4 = g_{\mu\nu} g^{\mu\nu} = 1 + g_{ij} g^{ij} = 1 + \delta_{ij} \delta^{ij} \,,$$

so that $\delta_{ij}\delta^{ij} = 3$.

### Integrals

For integrals, we define:

$$\int_x = \int \mathrm{d}^4 x \,, \quad \int_p = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \,, \quad \int_{\boldsymbol{p}} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \,, \quad \int_{|\boldsymbol{p}|} = \frac{1}{(2\pi)^2} \int_0^\infty \mathrm{d}|\boldsymbol{p}| |\boldsymbol{p}|^2 \,,$$

so that

$$\int_{\boldsymbol{p}} f(|\boldsymbol{p}|) = \int_{|\boldsymbol{p}|} f(|\boldsymbol{p}|)$$

for an isotropic function $f$, i. e. for a function which depends only on $|\boldsymbol{p}|$.

### Fourier Transformation

We define the (complete) Fourier transform of a function $f$ as

$$(\mathcal{F} f)(p) = \int \mathrm{d}^4 x \, f(x) \, \mathrm{e}^{\mathrm{i}\, p \cdot x} \,.$$

Correspondingly, the (complete) inverse Fourier transform is given by:

$$(\mathcal{F}^{-1} f)(x) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \, f(p) \, \mathrm{e}^{-\mathrm{i}\, p \cdot x} \,,$$

so that $\mathcal{F}^{-1}\mathcal{F} f = f$, or $\mathcal{F}^{-1}\mathcal{F} = \mathrm{id}$. In order not to clutter the notation, we will usually use the same symbol for a function $f$ and its Fourier transform $\mathcal{F} f$.

Changing from position space to momentum space then leads to the replacement rule

$$\partial_\mu \to -\mathrm{i}\, p_\mu \,.$$

We will also often need the partial Fourier transform of a function $f$ with respect to space,

$$f(x^0; \boldsymbol{p}) = \int \mathrm{d}^3 x \, f(x^0, \boldsymbol{x}) \, \mathrm{e}^{-\mathrm{i}\, \boldsymbol{p} \cdot \boldsymbol{x}} \,.$$

For an arbitrary spatially homogeneous two-point function $\mathcal{C}$, we then have

$$\mathcal{C}(x, y) = \mathcal{C}(x^0, y^0; \boldsymbol{x} - \boldsymbol{y}) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \, \mathcal{C}(x^0, y^0; \boldsymbol{p}) \, \mathrm{e}^{\mathrm{i}\, \boldsymbol{p} \cdot \boldsymbol{x}} \,,$$

so that it will be convenient to work with its partial Fourier transform $\mathcal{C}(x^0, y^0; \boldsymbol{p})$. Changing from position space to time–spatial-momentum space then leads to the replacement rules

$$\partial_{x\mu} \to \delta_\mu^0 \frac{\partial}{\partial x^0} - \mathrm{i}\, \delta_\mu^i p_i \,, \qquad \partial_{y\mu} \to \delta_\mu^0 \frac{\partial}{\partial y^0} + \mathrm{i}\, \delta_\mu^i p_i \,.$$

If we are only interested in the partially Fourier transformed quantity, we will then also often use the notation $(t, t') = (x^0, y^0)$, so that it reads $\mathcal{C}(t, t'; \boldsymbol{p})$.

### Symbols

The following table collects various other symbols used throughout this work.

| Symbol | Explanation |
|---|---|
| $\mu, \nu, \ldots$ | Lorentz (spacetime) indices (denoted by Greek letters) |
| $i, j, \ldots$ | spatial indices (denoted by Latin letters) |
| $:=$ | the left-hand side is defined to be equal to the right-hand side |
| $\partial_\mu = \partial_{x\mu} = \partial/\partial x^\mu$ | partial derivative |
| $\Box = g^{\mu\nu}\partial_\mu\partial_\nu$ | d'Alembertian |
| $[\cdot, \cdot]$ | commutator; $[f, g] = f\,g - g\,f$ |
| $\{\cdot, \cdot\}$ | anticommutator; $\{f, g\} = f\,g + g\,f$ |
| $\mathbf{1}$ | the unity matrix (where the number of dimensions can be inferred from the context) |
| $\gamma^\mu$ | gamma matrix; $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ |
| $z^*$ | complex conjugate of a complex number $z$ |
| $M^\dagger$ | Hermitean conjugate of a complex matrix $M$ |
| $\overline{M} = M^\dagger\gamma^0$ | Dirac conjugate of a complex matrix $M$ |
| Re, Im | real and imaginary parts of a complex number |
| U(1) | unitary group of degree 1; $\mathrm{U}(1) \cong \{z \in \mathbb{C} \,\vert\, z^* z = 1\} = \{\mathrm{e}^{\mathrm{i}\varphi} \,\vert\, \varphi \in \mathbb{R}\}$ |
| SU($n$) | special unitary group of degree $n$; $\mathrm{SU}(n) \cong \{M \in \mathrm{Mat}_n(\mathbb{C}) \,\vert\, M^\dagger M = \mathbf{1}\}$ where $\mathrm{Mat}_n(\mathbb{C})$ is the set of all $(n \times n)$ complex matrices |
| $m^{(\mathrm{f})}$ | fermion mass |
| $D_{\mu\nu}$ | photon (Feynman, i. e. time-ordered) propagator |
| $S$ | fermion (Feynman, i. e. time-ordered) propagator |
| $\Pi^{\mu\nu}$, $\Pi^{\mu\nu}_{(\rho)}$, $\Pi^{\mu\nu}_{(F)}$ | photon self-energy and its spectral and statistical parts |
| $\Sigma$, $\Sigma_{(\rho)}$, $\Sigma_{(F)}$ | fermion self-energy and its spectral and statistical parts |
| $f^{(\mathrm{g})}$ | generic photonic quantity |
| $f^{(\mathrm{f})}$ | generic fermionic quantity |
| $n_{\mathrm{BE}}(E)$ | Bose–Einstein distribution at inverse temperature $\beta$ |
| $n_{\mathrm{FD}}(E)$ | Fermi–Dirac distribution at inverse temperature $\beta$ |
| $\mathcal{A}_\mu$ | photon quantum field operator |
| $\Psi$ | (Dirac) fermion quantum field operator |

| | |
|---|---|
| $A_\mu$ | classical photon field or dummy variable in path integral |
| $\psi$ | classical fermion field or dummy variable in path integral |
| $F_{\mu\nu}$ | electromagnetic field strength |
| $B$ | Nakanishi–Lautrup field |
| $e$ | gauge coupling constant |
| $\Lambda$ | gauge parameter/function |
| $\xi$ | gauge fixing parameter |
| $Z$ | generating functional of correlation functions |
| $W$ | generating functional of connected correlation functions |
| $\Gamma_{1\text{PI}}$ | generating functional of 1PI correlation functions (proper vertex functions); (1PI) effective action |
| $\Gamma_{2\text{PI}}$ | generating functional of 2PI correlation functions; 2PI effective action |
| $\Gamma_2$ | 2PI part of the 2PI effective action |
| tr | trace over Dirac indices; $\text{tr}(A) = \sum_{a=1}^{4} A_{aa}$ |
| Tr | functional trace over spacetime arguments and Dirac indices if applicable; $\text{Tr}(A) = \int_x \sum_{a=1}^{4} A_{aa}(x,x)$ |

## Abbreviations

The following table explains the abbreviations used throughout this work.

| Abbreviation | Meaning |
|---|---|
| QED | Quantum Electrodynamics |
| QCD | Quantum Chromodynamics |
| QGP | Quark–Gluon Plasma |
| 1PI | One Particle Irreducible |
| 2PI | Two Particle Irreducible |
| EOM | Equation of Motion |
| CTP | Closed Time Path |
| DSE | Dyson–Schwinger equation |
| NL | Nakanishi–Lautrup |
| KMS | Kubo–Martin–Schwinger |
| DOF | Degree of Freedom |

# Chapter 1

# Introduction

## 1.1 General Introduction

Many physical processes proceed out of thermal equilibrium. Important examples are the very early universe, in particular the phenomenon of *reheating* [KLS94] which supposedly took place immediately after cosmological inflation [Gut81, Lin83] came to an end and which is believed to be responsible for the creation of matter; the thermalization of the quark-gluon plasma (QGP) produced in the "Little Bangs" [Hei01] of heavy ion collisions; and ultracold atomic gases [GBSS05, BG07, BG08, KG11]. It is clear that studying processes which start from a state away from thermal equilibrium means studying their time evolution since the nonexistence of time-translation invariance is what discriminates out-of-equilibrium processes from processes taking place in thermal equilibrium or, as a special case of equilibrium, in vacuum.

Due to the ubiquity of nonequilibrium processes in physics, the development of methods for dealing with such processes has started a long time ago. One important early example is the *Boltzmann equation* which governs the time evolution of a particle distribution function, i. e. of a function which describes the distribution of on-shell degrees of freedom (DOFs) as a function of time, position and momentum.[1] This corresponds to a kinetic description and is the basis of *kinetic theory*. This approach usually works very well in situations where the classical aspects of a physical system are dominant, like in a dilute gas where the de Broglie wavelength of the molecules or atoms the respective gas consists of is much shorter than the inter-particle distance so that there is a clear notion of "particle"[2] and hence of a distribution of particles.

If one integrates out the momentum dependence in the Boltzmann equation, one obtains an equation for the particle density, i. e. for the number of particles contained in a given volume. The corresponding equation then governs the dependence of the particle density on space and time, and, more general, *transport equations* describe the evolution of related

[1]One could also say that the Boltzmann equation describes the evolution of a particle distribution in phase space.

[2]Or at least of "quasiparticle", i. e. of something which behaves like a particle. Examples of quasiparticles are *phonons* in a solid or *plasmons* (see Sec. 3.2) in a thermal bath.

quantities like currents. One has thereby arrived at a *hydrodynamic* description of the system. This description has for instance turned out to be well-suited for experimental data regarding the QGP which behaves almost like an ideal fluid [Zaj08, Hei05]. For a hydrodynamic description to be possible, at least *locally* thermal equilibrium has to be given (see, however, Ref. [BBW04]).

Another approach is *linear response theory*, i. e. the theory of the relaxation of systems which are close to thermal equilibrium. The system can then be expanded around its equilibrium state, and quantities calculated in thermal equilibrium can be used to study its relaxation to thermal equilibrium. It is clear, however, that this can only possibly work for small deviations from thermal equilibrium. It is therefore suited for studying systems in thermal equilibrium which are slightly disturbed and pushed out of equilibrium, and their successive return to thermal equilibrium.

Mean field approaches also have a long history [CKMP95, CHK$^+$94, CHKM97]. However, they imply an infinite number of unphysical conserved quantities [Ber05] which severely restricts their domain of application.

Classical statistical simulations can be employed as a very good approximation to the dynamics of quantum systems if the occupation numbers of the system in question are large such that it behaves classically [AB02, BSS08, BH09]. It is clear, however, that this description breaks down e. g. if one aims at the thermalization of a quantum theory since a classical approximation will certainly fail to evolve a quantum system to (quantum) thermal equilibrium, i. e. to a Bose–Einstein or Fermi–Dirac distribution for bosons or fermions, respectively.

In conclusion, all methods mentioned so far for dealing with physical processes out-of-equilibrium have limitations or a restricted range of validity or applicability. It is therefore desirable to have a method at hand which on the one hand fully includes quantum effects of the system in question (i. e. which allows to study the time evolution not of classical fields like particle distribution functions in a Boltzmann approach or hydrodynamic modes in systems which are in thermal equilibrium locally, but of quantum fields), and on the other hand is able to (at least in principle) handle systems which at a given time are in a state far away from thermal equilibrium.

Techniques based on the *two-particle irreducible (2PI) effective action* [CJT74, Bay62, LW60] have turned out to be very well-suited for describing quantum fields out-of-equilibrium [CH88, Ber05].[3] With these methods, much progress has been achieved in recent years by simulating quantum fields out-of-equilibrium numerically. For instance, a quantum field theory of fermion production after inflation has been established in order to tackle the problem of matter creation mentioned earlier [BPR09, BGP11]. Further, thermalization of initially (highly) non-thermal states could be demonstrated numerically for scalar theories [BC01, AB01, Ber02] as well as for fermionic theories [BBS03]. So far, however, it has not been possible to show the thermalization of gauge theories as well. This is because

[3]The range of applicability of 2PI techniques is, however, not at all restricted to out-of-equilibrium or time evolution problems. For instance, they can be employed to study transport properties [AMR05] or bound states within approaches based on the Bethe–Salpeter equation [SB51].

there are intricate problems in the real-time formulation which are inherent to gauge theories and do not occur in non-gauge theories. These problems are all in one way or another related to the fact that there are non-physical DoFs present in covariant formulations of gauge theories. It is clear that this fact is unavoidable: A covariant formulation requires us to describe the gauge boson field by a four-(co)vector, but the gauge boson has only two fundamental DoFs, corresponding to the two possible polarization directions (spin states) of a massless particle (i. e. of a particle propagating with the speed of light).

While in vacuum, where one can work in momentum space, it is usually simple to project onto the physical DoFs, it will turn out that this is a severe problem in a real-time formulation. Instead of discarding the unphysical DoFs altogether in the first place, one has to keep and evolve them as well, and only after the time evolution has finished one can try to extract the physical DoFs.

Another problem is that of gauge invariance: Although the exact theory, with all its information encoded in the 2PI effective action, has to be gauge invariant, this need not be true for finite approximations of the 2PI effective action and correspondingly for quantities derived from it. This potential gauge non-invariance manifests itself in a dependence on the gauge fixing parameter, which can, depending on its value, render the approximation arbitrarily bad.

Foundational work regarding gauge theories within the 2PI framework has been carried out e. g. in Refs. [Ber04, Cal04, RS10]. In this work, however, we are mostly interested in a practical real-time formulation of gauge theories which is suitable for carrying out numerical simulations. We will restrict ourselves to the simplest gauge theory, namely QED. The great advantage of QED is that it is an abelian gauge theory, i. e. the gauge bosons do not exhibit self-interaction. On the other hand, however, fermions necessarily have to be included in order to obtain an interacting theory, which can create problems in their own right.

## 1.2 Outline of this Work

We start by introducing QED, the theory we will be concerned with in this work. We discuss the characteristic feature of gauge theories, namely their invariance under (local) gauge transformations, and the problems gauge symmetry causes when trying to set up a path integral in order to quantize a gauge theory. We promote classical electrodynamics to its quantum field theory (QFT), QED, by explicitly constructing its path integral representation.

We then come to the second important aspect of this work, namely nonequilibrium QFT. We will introduce the 2PI effective action which is a powerful tool for dealing with nonequilibrium QFT.

After that, from the 2PI effective action for QED we derive EOMs for the photon and fermion propagators, which contain important information for instance for questions regarding the thermalization of the theory, in a form which is suitable for studying their time evolution. It will turn out that in particular the photon EOMs are structurally rather complicated. We then discuss the question of DOFs, in particular for the photons. Since one important feature of a Lorentz covariant description of gauge theories is that it necessarily includes redundant DOFs, one has to face the question of which DOFs are physical. It will turn out that the answer to this question is much more involved in real-time formulations of gauge theories than in momentum-space formulations which can e. g. be employed in vacuum or thermal equilibrium where time-translation invariance is given. This is one example for a qualitative difference of real-time formulations in contrast to momentum-space formulations of gauge theories.

Due to the complicated form of the photon EOMs, we then propose a reformulation of the photon EOMs which is based on the introduction of an auxiliary field, the so-called *Nakanishi–Lautrup field*. For the free theory as well as for the full theory (i. e. for an untruncated effective action), it turns out that the EOMs for the propagators involving the auxiliary field are free and therefore can be solved exactly analytically. Since they also appear in the EOM for the pure photon propagator, their solutions can be plugged in, and they are then effectively "integrated out", i. e. one is left with the EOMs for the pure photon propagator only. The interesting point is that the resulting photon EOMs are structurally much simpler than the original ones, and they seem to be the "natural" formulation. Although instructive, however, this reformulation is of little practical value since in practice, one is of course only interested in an interacting theory, and its corresponding effective action has to be finitely truncated for concrete applications. The EOMs for the auxiliary field propagators are then not free, and one has effectively increased the number of DOFs and correspondingly of EOMs to solve. Nevertheless, the reformulation is convenient for obtaining the solutions to the free photon EOMs, which is easy compared to the original formulation of the EOMs. After providing the initial conditions for the photon and fermion correlation functions, which are necessary in order to solve their EOMs which are differential equations with respect to time, we then explicitly solve the free photon EOMs.

Their free solutions exhibit a very peculiar feature: They diverge in time, i. e. are *secular*. Although it is at first sight unexpected to have solutions which grow without bound, upon a closer examination it turns out that this is another manifestation of the fact that unphysical DOFs are present in gauge theories, and their behavior is *a priori* unpredictable. In fact, due to the problems mentioned above in discarding the unphysical DOFs in the first place as one can often do in momentum-space formulations of gauge theories, one practically has no choice but to evolve unphysical DOFs as well. This is an important characteristic of real-time formulations of gauge theories. We also discuss the origin of the secularities. For this, the reformulation of the photon EOMs again turns out to be valuable.

We then come to another important feature of real-time formulations of gauge theories, namely possible gauge dependencies of quantities which in the exact theory are known to be gauge invariant. This question is closely related to the question of the applicability of the Ward identities which relate correlation functions of different order in gauge theories. Due to the complicated resummation the 2PI effective action implements, it turns out that some nice features of perturbative approaches are lost. For instance, since the 2PI effective action mixes different perturbative orders, finitely truncated 2PI effective actions in general yield gauge dependencies which would not be present in a 1PI formulation.

After that, we come to the numerical implementation of the 2PI EOMs. We introduce a concrete truncation of the 2PI effective action and derive the self-energies from it. We then cast the equations in a form which is suitable for the implementation on a computer. Due to the structural complexity of the photon EOMs and the general complexity of the system, a numerical time evolution turns out to be very challenging. We conclude by presenting some results of a time evolution starting from nonequilibrium initial conditions.

Finally, there are appendices on the auxiliary field in the operator formalism; on gauge invariant quantities; on the generalized convolutions used in the calculation of the self-energies; on general properties of the two-point functions appearing in the EOMs; and on details of the numerical implementation.

A publication of the results presented in this work is in progress.

# Chapter 2

# Theoretical Background

In this chapter we present the theoretical background of this work. We start by recollecting the basic features of (classical) electrodynamics with a special focus on its invariance under (local) gauge transformations and by deriving a path integral which promotes classical electrodynamics to the quantum field theory of QED.

After that, we introduce the second important concept of this work, namely nonequilibrium QFT. It will turn out that, instead of the 1PI effective action following from the usual path integral to be introduced in the next section, the 2PI effective action naturally incorporates essential features of nonequilibrium QFT and is therefore a most valuable tool for studying quantum fields out-of-equilibrium. We will hence derive the 2PI effective action in some detail.

## 2.1 Quantum Electrodynamics

Since the theory which this work is build upon is QED, we start by introducing the theory and its basic features in this section.

### 2.1.1 Classical Action

The classical action for QED with the photon field $A_\mu$, the fermion field $\psi$,[1] and its Dirac conjugate $\overline{\psi} = \psi^\dagger \gamma^0$,

$$S[A, \overline{\psi}, \psi] = S_{\mathrm{g}}[A] + S_{\mathrm{f}}[\overline{\psi}, \psi] + S_{\mathrm{int}}[A, \overline{\psi}, \psi] \,, \tag{2.1}$$

consists of three parts. The pure gauge part is given by:

$$S_{\mathrm{g}}[A] = -\frac{1}{4} \int_x F_{\mu\nu} F^{\mu\nu} \,, \tag{2.2}$$

[1] The fermion can usually be interpreted as an electron; for the sake of generality, however, we will stick to “fermion”.

which only depends on the photon field $A_\mu$. Here, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor.[2] The fermion part reads

$$S_\mathrm{f}[\overline{\psi}, \psi] = \int_x \overline{\psi}\,(\gamma^\mu \partial_\mu - m^{(\mathrm{f})})\,\psi\,, \tag{2.3}$$

and the interaction part is given by:

$$S_\mathrm{int}[A, \overline{\psi}, \psi] = -e \int_x A_\mu \overline{\psi} \gamma^\mu \psi\,. \tag{2.4}$$

While the gauge part depends only on the photon field and the fermion part only on the fermion field, photon and fermion fields are coupled by the interaction part, and the strength of the coupling is determined by the coupling constant $e$, which is nothing but the electric charge.[3] The QED interaction is a vector-type interaction since it couples the photon (co-)vector to the vector current $J^\mu = e\,\overline{\psi}\gamma^\mu\psi$.

### 2.1.2 Gauge Transformations

In this subsection, we will consider gauge transformations. We will have to discriminate two kinds of gauge transformations, namely global and local ones. From a practical point of view, the only difference is that the first one acts in the same way at each point in spacetime, while the second one can act independently at each point in spacetime. Although seemingly a small difference, it is the second kind of gauge transformation which distinguishes gauge theories from other theories in a qualitative way.

#### Global Gauge Transformations

It can easily be verified that the action $S[A, \overline{\psi}, \psi]$ is invariant under the global gauge transformation

$$\begin{aligned} A_\mu(x) &\mapsto A_\mu^g(x) = A_\mu(x)\,, \\ \psi(x) &\mapsto \psi^g(x) = g\,\psi(x)\,, \\ \overline{\psi}(x) &\mapsto \overline{\psi}^g(x) = \overline{\psi}(x)\,g^{-1} \end{aligned} \tag{2.5}$$

with $g \in \mathrm{U}(1)$[4], since each Dirac bilinear $\overline{\psi}M\psi$ (with an arbitrary complex $(4\times4)$-matrix $M$) is clearly invariant under this transformation. The transformation is global

[2] Upon choosing a rest frame, one can identify the electric field as $E_i = F_{i0}$ and the magnetic field as $B_i = \varepsilon_i{}^{jk} F_{jk}$. In terms of electric and magnetic fields, the pure gauge part of the action reads $S_\mathrm{g}[\boldsymbol{E}, \boldsymbol{B}] = \int_x (\boldsymbol{E}^2 - \boldsymbol{B}^2)/2$.

[3] Its physical value at low energies is given by $e \approx 1/137$ and is therefore extremely small: QED is a very weakly coupled theory.

[4] Any group element $g$ can be parametrized by a real number $\Lambda$ as $g = \mathrm{e}^{\mathrm{i}\,\Lambda}$. Since group elements of U(1) are (commuting) numbers, left and right action by them are identical. We nevertheless prefer to write the action of a group element on a Dirac conjugate fermion as multiplication from the right to make it conform to the more general case of the action of elements of non-abelian gauge groups.

since it acts in the same way at each point in spacetime—$g$ is constant. It acts as the identity transformation on the photon field[5] (i. e. leaves it unchanged) and multiplies the fermion field by a phase, i. e. by an element of the group $\mathrm{U}(1)$.[6]

It is often convenient to consider infinitesimal transformations only, i. e. transformations with an infinitesimal parameter whose square vanishes. This amounts to not operating with the respective symmetry group, but with its Lie algebra. The infinitesimal form of the global gauge transformation (2.5) is given by

$$\begin{aligned} \delta_\Lambda A_\mu(x) &= 0\,, \\ \delta_\Lambda \psi(x) &= \mathrm{i}\,\Lambda\,\psi(x)\,, \\ \delta_\Lambda \overline{\psi}(x) &= -\,\mathrm{i}\,\overline{\psi}(x)\Lambda\,. \end{aligned} \tag{2.6}$$

Since a global gauge transformation is a continuous transformation (the gauge parameter is an element of the Lie group of $\mathrm{U}(1)$ which is isomorphic to the real numbers, i. e. $\Lambda \in \mathrm{Lie}(\mathrm{U}(1)) \cong \mathbb{R}$), the symmetry of the classical action (2.1) under this transformations implies the existence of a conserved current via Noether's theorem. This conserved current is just what the photon field couples to in the interaction, i. e. $J^\mu = e\,\overline{\psi}\gamma^\mu\psi$, with $\partial_\mu J^\mu = 0$.

### Local Gauge Transformations

Much more important consequences follow however from the invariance of QED under *local* gauge transformations.[7] We promote the global gauge transformation (2.5) to a local one by assuming the gauge parameter to be dependent on spacetime, thereby turning it into a gauge *function*. It is easy to see, however, that upon assuming the gauge function to depend on spacetime, the transformation (2.5) ceases to be a symmetry of the classical action (2.1). This is so because of the kinetic term in the fermion part of the action, which contains a derivative which acts on the gauge function and hence generates an additional term. We have:

$$\begin{aligned} S_\mathrm{f}[\overline{\psi},\psi] \mapsto S_\mathrm{f}[\overline{\psi}^\Lambda,\psi^\Lambda] &= \int_x \overline{\psi}(x)\,\mathrm{e}^{-\,\mathrm{i}\,\Lambda(x)}(\mathrm{i}\,\gamma^\mu\partial_\mu - m^{(\mathrm{f})})\,\mathrm{e}^{\mathrm{i}\,\Lambda(x)}\,\psi(x) \\ &= S_\mathrm{f}[\overline{\psi},\psi] - \frac{1}{e}\int_x \big[\partial_\mu\Lambda(x)\big]J^\mu(x)\,. \end{aligned} \tag{2.7}$$

In order to obtain an action which is invariant also under local gauge transformations, the transformation of the photon field has to be modified. In fact, the classical action is

[5]In fact, gauge fields transform in the adjoint representation,

$$A_\mu \mapsto A_\mu^g = g\,A_\mu\,g^{-1}\,.$$

Since for abelian gauge groups, $g$ and $A_\mu$ are (commuting) numbers, a transformation in the adjoint representation of an abelian group always acts like the identity transformation.

[6]To be more precise, the fermion field transforms in the fundamental representation of $\mathrm{U}(1)$, while its Dirac conjugate transforms in the antifundamental representation.

[7]If not otherwise stated, by "gauge transformation" we always mean *local* gauge transformations from now on.

invariant under the local gauge transformation

$$\begin{aligned} A_\mu(x) &\mapsto A^g_\mu(x) = g(x)A_\mu(x)\, g^{-1}(x) - \frac{\mathrm{i}}{e}\, g(x)\partial_\mu\, g^{-1}(x)\,, \\ \psi(x) &\mapsto \psi^g(x) = g(x)\,\psi(x)\,, \\ \overline{\psi}(x) &\mapsto \overline{\psi}^g(x) = \overline{\psi}(x)\, g^{-1}(x)\,, \end{aligned} \tag{2.8}$$

or in infinitesimal form:

$$\begin{aligned} \delta_\Lambda A_\mu(x) &= -\frac{1}{e}\,\partial_\mu\Lambda(x)\,, \\ \delta_\Lambda\psi(x) &= \mathrm{i}\,\Lambda(x)\,\psi(x)\,, \\ \delta_\Lambda\overline{\psi}(x) &= -\,\mathrm{i}\,\overline{\psi}(x)\Lambda(x)\,. \end{aligned} \tag{2.9}$$

The pure gauge part is still separately invariant under the local gauge transformation (due to the antisymmetry of the electromagnetic field strength tensor $F_{\mu\nu}$, the terms containing the gauge function cancel[8]), but the interaction part is not due to the nontrivial transformation of the photon field. In fact, the transformation of the interaction,

$$\begin{aligned} S_{\mathrm{int}}[A,\overline{\psi},\psi] \mapsto S_{\mathrm{int}}[A^\Lambda,\overline{\psi}^\Lambda,\psi^\Lambda] &= -e\int_x \Big[A_\mu(x) - \frac{1}{e}\,\partial_\mu(x)\Lambda(x)\Big]\,\overline{\psi}(x)\gamma^\mu\psi(x) \\ &= S_{\mathrm{int}}[A,\overline{\psi},\psi] + \frac{1}{e}\int_x \Big[\partial_\mu\Lambda(x)\Big]\,J^\mu(x)\,, \end{aligned} \tag{2.10}$$

exactly cancels the additional part in the transformation of the fermion part (2.7). Note that it reduces to the global gauge transformation (2.5) in the case of a constant gauge function, $\partial_\mu\Lambda(x) = 0$.

Further note that, although the purely fermionic sector of QED is invariant under the global gauge transformation (2.5) on its own, in order to obtain invariance under local gauge transformations, one is forced to introduce the gauge sector as well in order to compensate for the noninvariance of the purely fermionic sector under local gauge transformations.

It is convenient to introduce the *gauge covariant derivative* $\mathrm{D}_\mu = \partial_\mu + \mathrm{i}\,eA_\mu$. It cancels the additional term generated by the partial derivative in the kinetic fermion term, so that under a gauge transformation,

$$\mathrm{D}_\mu(x) \mapsto \mathrm{D}^g_\mu(x) = g(x)\mathrm{D}_\mu(x)g^{-1}(x)\,,$$

and therefore

$$\mathrm{D}_\mu(x)\psi(x) \mapsto \mathrm{D}^g_\mu(x)\psi^g(x) = g(x)\mathrm{D}_\mu(x)\psi(x)\,.$$

In contrast to the partial derivative of a Dirac spinor, which does not transform (gauge) covariantly due to the additional term containing the gradient of the gauge function, the

[8]This is very easy to see in the language of differential forms. In an index-free notation, the electromagnetic field strength is given by $F = \mathrm{d}A$, so that under a gauge transformation $F = \mathrm{d}A \mapsto \mathrm{d}(A - \mathrm{d}\Lambda) = \mathrm{d}A - \mathrm{d}^2\Lambda = \mathrm{d}A = F$, where we have absorbed the electric charge into the gauge function $\Lambda$ and used that $\mathrm{d}^2 = 0$.

gauge covariant derivative of a Dirac spinor does transform covariantly, i. e. in the same way as a Dirac spinor itself. It immediately follows that $\overline{\psi}(x)\mathrm{D}_\mu(x)\psi(x)$ is gauge invariant, and therefore

$$S_{\mathrm{f}}[\overline{\psi},\psi] + S_{\mathrm{int}}[A,\overline{\psi},\psi] = \int_x \overline{\psi}(\mathrm{i}\,\gamma^\mu \mathrm{D}_\mu - m^{(\mathrm{f})})\psi$$

is gauge invariant as well. Note that the term containing the photon field in the gauge covariant derivative exactly generates the interaction term.

An infinitesimal gauge transformation of the complete action then reads:

$$\begin{aligned}
&\delta_\Lambda S[A,\overline{\psi},\psi] \\
&\quad= \int_x \delta\Lambda(x)\,\frac{\delta_\Lambda S[A,\overline{\psi},\psi]}{\delta\Lambda(x)} \\
&\quad= \int_x \left[\delta_\Lambda A_\mu(x)\,\frac{\delta}{\delta A_\mu(x)} + \delta_\Lambda\psi(x)\,\frac{\delta}{\delta\psi(x)} + \delta_\Lambda\overline{\psi}(x)\,\frac{\delta}{\delta\overline{\psi}(x)}\right] S[A,\overline{\psi},\psi] \\
&\quad= \int_x \left\{-\frac{1}{e}\big[\partial_\mu\Lambda(x)\big]\,\frac{\delta}{\delta A_\mu(x)} + \mathrm{i}\,\Lambda(x)\psi(x)\,\frac{\delta}{\delta\psi(x)} - \mathrm{i}\,\Lambda(x)\overline{\psi}(x)\,\frac{\delta}{\delta\overline{\psi}(x)}\right\} S[A,\overline{\psi},\psi] \\
&\quad= \int_x \Lambda(x)\left[\frac{1}{e}\,\partial_\mu\,\frac{\delta}{\delta A_\mu(x)} + \mathrm{i}\,\psi(x)\,\frac{\delta}{\delta\psi(x)} - \mathrm{i}\,\overline{\psi}(x)\,\frac{\delta}{\delta\overline{\psi}(x)}\right] S[A,\overline{\psi},\psi] \\
&\quad= \int_x \Lambda(x)\,\mathcal{G}(x) S[A,\overline{\psi},\psi]\,, 
\end{aligned} \tag{2.11}$$

where in the last line we have defined the *generator of gauge transformations*[9]

$$\mathcal{G}(x) = \frac{1}{e}\,\partial_\mu\,\frac{\delta}{\delta A_\mu(x)} + \mathrm{i}\,\psi(x)\,\frac{\delta}{\delta\psi(x)} - \mathrm{i}\,\overline{\psi}(x)\,\frac{\delta}{\delta\overline{\psi}(x)}\,. \tag{2.12}$$

### 2.1.3 Classical Effective Action

In this subsection, we will quantize classical electrodynamics by setting up a path integral for it. We will find that contrary to non-gauge theories, where the classical action appears in the path integral, in gauge theories it is not the classical action of the corresponding classical theory itself which appears in the path integral, but a modified action which we call "classical effective action" and which we shall derive shortly. First, however, in order to become familiar with the underlying concepts, we will briefly consider the path integral quantization of a scalar theory.

#### The Concept of the Generating Functional of Correlation Functions

A convenient way to quantize a theory is to set up a path integral which allows for the definition of correlation functions, which encode all the information contained in the respective QFT. For a scalar theory with action $S[\phi]$, the path integral, which is called the

[9] It is essentially a translation along the *gauge orbit* (see Sec. 2.1.3). It can also be written as $\mathcal{G}(x) = \{\phi(x),\cdot\}_{\mathrm{PB}}$, where $\phi(x) = \partial_i F^{i0}(x) - J^0(x) = \boldsymbol{\nabla}\cdot\boldsymbol{E}(x) - \rho(x) = 0$ is the Gauss constraint and $\{\cdot,\cdot\}_{\mathrm{PB}}$ is the Poisson bracket. It is hence the Gauss constraint which generates gauge transformations. It follows that a functional $\mathcal{F}[A,\overline{\psi},\psi]$ is gauge invariant if $\mathcal{G}(x)\mathcal{F}[A,\overline{\psi},\psi] = \{\phi(x),\mathcal{F}[A,\overline{\psi},\psi]\}_{\mathrm{PB}} = 0$.

*generating functional (of correlation functions)*, is given by

$$Z[J] = \mathcal{N} \int \mathcal{D}\phi \, \exp\Big(\mathrm{i}\Big\{S[\phi] + \int_x J(x)\,\phi(x)\Big\}\Big) \,, \tag{2.13}$$

where $\mathcal{N}$ is some irrelevant normalization constant and $\mathcal{D}\phi$ is the integration measure.[10] Note that $\phi$ is not a quantum field, but a classical field which can take on arbitrary configurations (it is an integration variable[11], i. e. a "dummy field"). The classical external source $J$ coupled linearly to the field is introduced for computational purposes since it allows for an easy way to obtain correlation functions of the field. In fact, one has:

$$\begin{aligned}
\frac{\delta Z[J]}{\delta J(x)} &= \mathrm{i}\mathcal{N} \int \mathcal{D}\phi \left[\frac{\delta}{\delta J(x)} \int_y J(y)\,\phi(y)\right] \exp\Big(\mathrm{i}\Big\{S[\phi] + \int_x J(x)\,\phi(x)\Big\}\Big)\\
&= \mathrm{i}\mathcal{N} \int \mathcal{D}\phi \int_y \delta^4(y-x)\,\phi(y) \exp\Big(\mathrm{i}\Big\{S[\phi] + \int_x J(x)\,\phi(x)\Big\}\Big)\\
&= \mathrm{i}\mathcal{N} \int \mathcal{D}\phi\, \phi(x) \exp\Big(\mathrm{i}\Big\{S[\phi] + \int_x J(x)\,\phi(x)\Big\}\Big)\\
&=: \mathrm{i}\langle\phi(x)\rangle_J \,.
\end{aligned} \tag{2.14}$$

Note that the last line is just notation made to resemble the expectation value of the corresponding quantum field, and the subscript is to remind that this is the "expectation value" in the presence of the external source $J$. Since it was only introduced for computational reasons and is not physical, it is usually set to zero at the end of the calculation, and we therefore define $\langle\phi(x)\rangle := \langle\phi(x)\rangle_0$.

By repeatedly applying derivatives of the generating functional with respect to the external source, we can therefore obtain any correlation function of the fields via

$$\langle\phi(x_1)\ldots\phi(x_n)\rangle_J = \frac{1}{\mathrm{i}^n} \frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)} \,, \tag{2.15}$$

hence the name "generating functional of correlation functions". Since the generating functional encodes all the information contained in the corresponding QFT, by Taylor expanding it,

$$\begin{aligned}
Z[J] &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{x_1,\ldots,x_n} \left.\frac{\delta^n Z[J]}{\delta J(x_1)\ldots\delta J(x_n)}\right|_{J=0} J(x_1)\ldots J(x_n)\\
&= \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} \int_{x_1,\ldots,x_n} \langle\phi(x_1)\ldots\phi(x_n)\rangle\, J(x_1)\ldots J(x_n)
\end{aligned} \tag{2.16}$$

[10] Properly defining a path integral is a highly nontrivial task, since the integration "variables" are fields which are infinite-dimensional objects. The integration measure is then something like $\mathcal{D}\phi = \prod_x \mathrm{d}\phi(x)$, where $\prod_x$ denotes a continuous product over all spacetime points. It is immediately clear, however, that a "continuous product" is not a simple object.

The difficulties in defining the path integral are, however, not so important for our concern since we are only interested in computing quantities from it.

[11] To be precise: an infinite number of integration variables, one at each point in spacetime.

it becomes clear that equivalently, all the information is contained in the (infinite number of) correlation functions of the fields.

It is important to note that in the definition of the generating functional, Eq. (2.13), the classical action appears in the exponential, since, as we shall see now, this is not the case for gauge theories.

### The Generating Functional for Gauge Theories

It is easy to see that the definition (2.13) for the generating functional of some scalar field theory cannot easily be carried over to gauge theories, i. e. it does *not* make sense to define, for instance, for QED:

$$\begin{aligned} &Z[J,\eta,\overline{\eta}] \\ &\quad = \mathcal{N} \int \mathcal{D}A\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi\, \exp\!\left( \mathrm{i} \left\{ S[A,\overline{\psi},\psi] + \int_x \Big[ A_\mu(x) J^\mu(x) + \overline{\psi}(x)\,\eta(x) + \overline{\eta}(x)\,\psi(x) \Big] \right\} \right), \end{aligned} \tag{2.17}$$

where $\mathcal{D}A = \prod_{\mu=0}^{3} \mathcal{D}A_\mu = \mathcal{D}A_0\,\mathcal{D}A_1\,\mathcal{D}A_2\,\mathcal{D}A_3$. The reason that this definition does not make sense is the gauge symmetry: Since quantities related by a gauge transformation are physically indistinguishable, the path integral overcounts the number of physical states. It is therefore important to only sum over physically distinct states in order to set up a well-defined path integral for gauge theories.[12]

Since this affects only the photon sector, let us focus on it and neglect the fermions for now (they can later be added to the generating functional in the usual way without causing any problems). Further, we will ignore the external source, which can always be added back at the end by simply coupling it to the photon field as in (2.17).

We would then like to be able to factor the "naive", overcounting generating functional into the "volume" of the gauge group and the part which only counts physically distinct states, i. e.

$$\widetilde{Z}[0] = \int \mathcal{D}A\, \exp(\mathrm{i}\, S[A]) = \left[ \int_{\mathrm{U}(1)} \mathcal{D}\mu(g) \right] Z[0] = V_{\mathrm{U}(1)}\, Z[0]\,,$$

where

$$\mathcal{D}\mu(g) := \prod_x \mathrm{d}\mu(g(x))$$

is the *Haar measure* of the gauge group, and the *Haar integral*

$$V_{\mathrm{U}(1)} := \int_{\mathrm{U}(1)} \mathcal{D}\mu(g)$$

[12] In fact, if we were only calculating gauge invariant quantities, the overcounting could be canceled by the normalization constant of the generating functional. However, it is usually not possible to get along without employing gauge noninvariant quantities, if only in intermediate steps of the calculation. In particular, this is not possible in perturbation theory which depends on the free inverse propagators which, as we shall see shortly, are not gauge invariant.

is the volume of the gauge group U(1).

First of all, we will need some concepts and notation with respect to groups. We denote the action of an element $g$ of a group to some quantity by attaching the element as a superscript to the quantity, e. g. $A_\mu^g$. Sometimes we will also write $A_\mu^\Lambda$ for $g(x) = \exp(\mathrm{i}\,\Lambda(x))$, i. e. attach the corresponding element of the respective Lie algebra.

Two gauge fields $A_\mu$ and $A'_\mu$ are called *(gauge) equivalent* if they are related by a gauge transformation, i. e.

$$A_\mu \sim A'_\mu \;\Leftrightarrow\; \exists\, g \in \mathrm{U}(1):\; A_\mu^g = A'_\mu \;\Leftrightarrow\; \exists\, \Lambda \in \mathbb{R}:\; A_\mu - \frac{1}{e}\,\partial_\mu \Lambda = A'_\mu\,.$$

For a given gauge field $A_\mu$, the set of all gauge fields which are equivalent to it forms an equivalence class which is called the *(gauge) orbit* of $A_\mu$,

$$[A_\mu] = \{A'_\mu \,|\, A'_\mu \sim A_\mu\}\,.$$

Each element of the orbit is a *representative* of that orbit. Gauge fields belonging to the gauge orbit of zero, [0], are called *pure gauge*. Note that all longitudinal gauge fields are pure gauge.

It is then clear that in the generating functional, we would like to include exactly one representative of each gauge orbit in the integral. We therefore need a way to pick one representative of each orbit, which is called *fixing a gauge* or just *gauge fixing*. For each gauge orbit, we therefore require the representative $A_\mu$ to satisfy an equation of the form $F(A) = 0$, where $F$ is some function which may include differential operators. The equation $F(A) = 0$ parametrizes a hypersurface in the space of gauge fields which should be intersected by each gauge orbit exactly once, and the intersection points of all gauge orbits fill the hypersurface defined implicitly by $F(A) = 0$ completely.[13] In other words: The generating functional should not be over the space of all gauge fields, but only over the hypersurface defined by $F(A) = 0$. The procedure to implement this condition is due to *Faddeev* and *Popov* [FP67], and we will briefly explain it in the following.

Assume that we would like to integrate some function $G$ of a fixed gauge field $A_\mu$ along its gauge orbit,

$$\tilde{I}(A) = \int_{\mathrm{U}(1)} \mathcal{D}\mu(g)\, G(A^g)\,.$$

Then in order to only take into account the contribution stemming from the intersection of the gauge orbit with the hypersurface defined by $F(A) = 0$, we simply include a delta distribution enforcing this condition:

$$\tilde{I}(A) \to I(A) = \Delta(A) \int_{\mathrm{U}(1)} \mathcal{D}\mu(g)\, \delta[F(A^g)]\, G(A^g)$$

with

$$\delta[F(A)] := \prod_x \delta(F(A(x)))\,,$$

[13] In practice, this condition may not be satisfiable due to the existence of the so-called *Gribov ambiguity* [Gri78].

where the normalization factor $\Delta(A)$ ensures that

$$\Delta(A) \int_{\mathrm{U}(1)} \mathcal{D}\mu(g)\, \delta[F(A^g)] = 1\,. \tag{2.18}$$

The normalization to unity is convenient since we can now just insert it into the generating functional without changing its value.[14] We obtain:

$$\begin{aligned}
\widetilde{Z}[0] &= \int \mathcal{D}A\, \Delta[A] \int_{\mathrm{U}(1)} \mathcal{D}\mu(g)\, \delta[F(A^g)] \exp(\mathrm{i}\, S[A]) \\
&= \int \mathcal{D}A^g\, \Delta[A^g] \int_{\mathrm{U}(1)} \mathcal{D}\mu(g)\, \delta[F[A^g]] \exp(\mathrm{i}\, S[A^g]) \\
&= \int \mathcal{D}A\, \Delta[A]\, \delta[F(A)] \exp(\mathrm{i}\, S[A]) \int_{\mathrm{U}(1)} \mathcal{D}\mu(g) \\
&= V_{\mathrm{U}(1)} \int \mathcal{D}A\, \Delta[A]\, \delta[F(A)] \exp(\mathrm{i}\, S[A])\,.
\end{aligned}$$

Here we have used that $\Delta[A]$ is gauge invariant, i. e. $\Delta[A] = \Delta[A^g]$. By definition, the Haar measure of the gauge group is gauge invariant as well, so that $\mathcal{D}(g'\, g) = \mathcal{D}g$, and hence

$$\int \mathcal{D}\mu(g)\, \delta[F(A^g)] \mapsto \int \mathcal{D}\mu(g)\, \delta[F(A^{g'g})] = \int \mathcal{D}\mu(g'g)\, \delta[F(A^{g'g})] = \int \mathcal{D}\mu(g)\, \delta[F(A^g)]\,.$$

Since 1 is clearly gauge invariant, so is $\Delta[A]$ according to Eq. (2.18). We therefore find

$$Z[0] = \frac{\widetilde{Z}[0]}{V_{\mathrm{U}(1)}} = \int \mathcal{D}A\, \Delta[A]\, \delta[F(A)]\, \exp(\mathrm{i}\, S[A])$$

as a meaningful definition of the (source-free) generating functional, i. e. as a definition which does not overcount physically equivalent states.

The next goal is to bring this into a form which is more easily manageable, i. e. to find more practical expressions for $\Delta[A]$ and $\delta[F(A)]$. Ideally, we would like to end up with an expression like

$$Z[0] = \int \mathcal{D}A\, \Delta[A]\, \delta[F(A)]\, \exp(\mathrm{i}\, S[A]) = \int \mathcal{D}A\, \exp(\mathrm{i}\, S_{\text{eff}}[A])$$

with some "effective" classical action $S_{\text{eff}}[A]$ which incorporates the effects of limiting the path integral to physically distinct states, since we could then treat the gauge theory described by the classical action $S[A]$ essentially in the same way as a non-gauge theory with classical action $S_{\text{eff}}[A]$.

We have:

$$\begin{aligned}
1 &= \int \mathcal{D}g\, \Delta[A^g]\, \delta[F(A^g)] = \int \mathcal{D}g\, \Delta[A^g] \left[\det\left(\frac{\delta F(A^g)}{\delta g}\right)\bigg|_{g=g_0}\right]^{-1} \delta[g - g_0] \\
&= \Delta[A^{g_0}] \left[\det\left(\frac{\delta F(A^g)}{\delta g}\right)\bigg|_{g=g_0}\right]^{-1} = \Delta[A] \left[\det\left(\frac{\delta F(A^g)}{\delta g}\right)\bigg|_{g=1}\right]^{-1}
\end{aligned}$$

[14] Note that it is not an overall normalization (which would be irrelevant and could be ignored) since it still depends on the fixed gauge field $A_\mu$ and hence contributes to the path integral in a nontrivial way.

where $F(A^{g_0}) = 0$, i.e. the gauge field which is obtained from the original fixed gauge field $A_\mu$ by a gauge transformation with $g_0$ is the one (and only one in its gauge orbit) which satisfies the gauge condition. We can, however, without loss of generality choose the original gauge field thus that it already satisfies the gauge condition, so that $g_0 = 1$. It follows that:

$$\Delta[A] = \det\left(\frac{\delta F(A^g)}{\delta g}\right)\Bigg|_{g=1} ,$$

where the argument of the determinant defines the so-called *Faddeev-Popov operator* $M$ with

$$M_A(x,y) := \frac{\delta F(A^g(x))}{\delta g(y)}\Bigg|_{g=1} = \int_z \frac{\delta F(A(x))}{\delta A_\mu(z)}\,\frac{\delta A_\mu^g(z)}{\delta g(y)}\Bigg|_{g=1} ,$$

while $\Delta[A] = \det(M_A)$ is correspondingly called the *Faddeev-Popov determinant*. The determinant can be represented by an exponential:

$$\begin{aligned}\det(M_A) &= \int \mathcal{D}\overline{c}\,\mathcal{D}c\,\exp\Big(-\int_{x,y}\overline{c}(x)M_A(x,y)c(y)\Big)\\ &= \int \mathcal{D}\overline{c}\,\mathcal{D}c\,\exp\Big(\mathrm{i}\int_{x,y}\overline{c}(x)\,\mathrm{i}\,M_A(x,y)c(y)\Big) ,\end{aligned}$$

where the (auxiliary, i.e. unphysical) fields $c$ and $\overline{c}$ are called *Faddeev-Popov ghosts*. We can then rewrite the generating functional as

$$Z[0] = \int \mathcal{D}A\,\mathcal{D}\overline{c}\,\mathcal{D}c\,\delta[F(A)]\,\exp\Big(\mathrm{i}\Big\{S[A] + \int_{x,y}\overline{c}(x)\,\mathrm{i}\,M_A(x,y)c(y)\Big\}\Big) .$$

We have therefore traded the nonlocal Faddeev-Popov determinant for the introduction of two new fields.

We can now make use of the fact that

$$\delta[F(A)] = \delta[F(A) - B] ,$$

which is true because $B$ is independent of $A$.[15] We then have:

$$Z[0] = \int \mathcal{D}A\,\mathcal{D}\overline{c}\,\mathcal{D}c\,\delta[F(A) - B]\,\exp\Big(\mathrm{i}\Big\{S[A] + \int_{x,y}\overline{c}(x)\,\mathrm{i}\,M_A(x,y)c(y)\Big\}\Big) .$$

[15] Compare this to a usual function $f$ with the single root $x_0$. One has

$$\delta(f(x)) = \frac{1}{|f'(x_0)|}\,\delta(x - x_0) .$$

Now consider the function $g$ with $g(x) = f(x) - c$ and assume it has the single root $\widetilde{x}_0$. Then:

$$\delta(g(x)) = \delta(f(x) - c) = \frac{1}{|f'(\widetilde{x}_0)|}\,\delta(x - x_0) ,$$

since $g' = f'$. It follows that

$$\delta(f(x)) = \left|\frac{f'(\widetilde{x}_0)}{f'(x_0)}\right|\delta(f(x) - c) = \widetilde{c}\,\delta(f(x) - c)$$

with the constant $\widetilde{c}$ (which is in particular independent of $x$).

It follows that $Z[0]$ is independent of $B$, so that, for an arbitrary real parameter $\xi$,

$$\begin{aligned} Z[0] = \mathcal{N} \int \mathcal{D}B \exp\left(-\frac{\mathrm{i}}{2\xi}\int_x B(x)^2\right) \\ \cdot \int \mathcal{D}A\,\mathcal{D}\overline{c}\,\mathcal{D}c\,\delta[F(A)-B]\,\exp\left(\mathrm{i}\left\{S[A]+\int_{x,y}\overline{c}(x)\,\mathrm{i}\,M_A(x,y)c(y)\right\}\right) \\ = \mathcal{N}\int \mathcal{D}A\,\mathcal{D}\overline{c}\,\mathcal{D}c\,\exp\left(\mathrm{i}\left\{S[A]-\frac{1}{2\xi}\int_x F(A(x))^2+\int_{x,y}\overline{c}(x)\,\mathrm{i}\,M_A(x,y)c(y)\right\}\right) \\ = \mathcal{N}\int \mathcal{D}A\,\mathcal{D}\overline{c}\,\mathcal{D}c\,\exp(\mathrm{i}\,S_{\mathrm{eff}}[A,\overline{c},c]) \end{aligned}$$

with the effective classical action

$$\begin{aligned} S_{\mathrm{eff}}[A,\overline{c},c] &= S_{\mathrm{g}}[A] - \frac{1}{2\xi}\int_x F(A(x))^2 + \int_{x,y}\overline{c}(x)M_A(x,y)c(y) \\ &= S_{\mathrm{g}}[A] + S_{\mathrm{gf}}[A] + S_{\mathrm{gh}}[A,\overline{c},c] \end{aligned} \tag{2.19}$$

with the gauge fixing term

$$S_{\mathrm{gf}}[A] = -\frac{1}{2\xi}\int_x F(A(x))^2 \tag{2.20}$$

and the ghost term

$$S_{\mathrm{gh}}[A,\overline{c},c] = \int_{x,y}\overline{c}(x)\,\mathrm{i}\,M_A(x,y)c(y)\,. \tag{2.21}$$

Note that the Faddeev-Popov operator is nonlocal in general (bilocal, to be more precise), and that the ghost term in general depends on the gauge field as well, i. e. the ghosts couple to the gauge field unless the Faddeev-Popov operator is linear in the gauge field.

Making use of the concrete form of the gauge transformation of the photon field, we can simplify the Faddeev-Popov operator somewhat. With $g(x) = \exp(\mathrm{i}\,\Lambda(x))$, the gauge transformation of the photon field obtained by acting with $g(x)$ on it is given by

$$A^g_\mu(x) = A_\mu(x) - \frac{1}{e}\,\partial_\mu\Lambda(x) = A_\mu(x) - \frac{1}{\mathrm{i}\,e}\,\partial_\mu\ln(g(x)) = A_\mu(x) - \frac{1}{\mathrm{i}\,e}\,\frac{\partial_\mu g(x)}{g(x)}\,.$$

It follows that

$$\begin{aligned} \left.\frac{\delta A^g_\mu(x)}{\delta g(y)}\right|_{g=1} &= \frac{\delta}{\delta g(y)}\left.\left[A_\mu(x) - \frac{1}{\mathrm{i}\,e}\,\frac{\partial_{x\mu}g(x)}{g(x)}\right]\right|_{g=1} \\ &= -\frac{1}{\mathrm{i}\,e}\left.\left[-\frac{1}{g(x)^2}\,\delta^4(x-y)\,\partial_{x\mu}g(x) + \frac{1}{g(x)}\,\partial_{x\mu}\delta^4(x-y)\right]\right|_{g=1} \\ &= -\frac{1}{\mathrm{i}\,e}\,\partial_{x\mu}\delta^4(x-y)\,, \end{aligned}$$

so that

$$\mathrm{i}\,M_A(x,y) = -\frac{1}{e}\int_z \frac{\delta F(A(x))}{\delta A_\mu(z)}\,\partial_{x\mu}\delta^4(z-y) = \frac{1}{e}\,\partial_{y\mu}\,\frac{\delta F(A(x))}{\delta A_\mu(y)}\,.$$

The ghost term can then be written as

$$S_{\text{gh}}[A,\bar{c},c] = \frac{1}{e}\int_{x,y} \bar{c}(x)\left[\partial_{y\mu}\,\frac{\delta F(A(x))}{\delta A_\mu(y)}\right] c(y)\,. \tag{2.22}$$

Note that if $F(A)$ is linear in $A_\mu$, then the Faddeev-Popov operator and hence the ghost term is independent of the gauge field. It is then often useful not to introduce the Faddeev-Popov ghosts but just keep the Faddeev-Popov determinant as an overall factor (which, of course, is only possible if it does not depend on the gauge field).

### Gauge Fixing

So far, we have not specified the function $F(A)$ which fixes the gauge. As is clear from what we have said above, the only sensible choice is a function which is linear in the gauge field, since then the ghosts decouple from the gauge field. A linear gauge fixing function can always be written in the form

$$F_{\text{linear}}(A) = f^\mu A_\mu$$

where $f^\mu$ is a collection of four quantities (not necessarily numbers and not necessarily forming a Lorentz vector), and the gauges specified by a linear gauge fixing function are correspondingly called *linear gauges*.

In vacuum, there is only one *a priori* or naturally given vector, namely the partial derivative. In that case, i. e. $f^\mu = \partial^\mu$, one has

$$F_{\text{covariant}}(A) = \partial^\mu A_\mu\,,$$

so that the gauge fixing function is Lorentz invariant (i. e. $F_{\text{covariant}}(A)$ is a Lorentz scalar), and the corresponding gauge is called *(linear) covariant gauge*.[16]

Since there is no other naturally given vector in vacuum, all gauges which do not belong to the class of covariant gauges are therefore called *noncovariant gauges*. An important class of noncovariant gauges is given by choosing $f^\mu = n^\mu$ where $n^\mu$ is a collection of four numbers. These gauges are therefore defined by the condition

$$F_{\text{axial}}(A) = n^\mu A_\mu\,.$$

$n^\mu$ constitutes a preferred direction or axis, and hence these gauges are called *axial gauges*.[17] Note that the gauge fixing function is not Lorentz invariant.

[16] Multiplying the gauge fixing function by some nonzero number does not change the gauge condition, so the gauge condition is essentially uniquely defined.

[17] One can further distinguish the axial gauges according to the nature of $n$: If $n$ is timelike, i. e. if $n^2 > 0$ (like, for instance, $n = (1,0,0,0)^\top$, so that $A_0 = 0$), one speaks of *temporal (axial) gauges*. If $n$ is spacelike, i. e. if $n^2 < 0$ (like, for instance, $n = (0,0,0,1)^\top$, so that $A_3 = 0$), one speaks of *(spatial) axial gauges*. And finally, if $n$ is lightlike, i. e. if $n^2 = 0$ (like, for instance, $n = (1,0,0,1)^\top$, so that $A_0 + A_3 = 0$), one speaks of *light cone gauges*.

Another important class of noncovariant gauges is given by choosing $f^\mu = (g^{\mu\nu} - n^\mu n^\nu)\partial_\nu = \partial^\mu - n^\mu n^\nu \partial_\nu$. The corresponding gauge fixing condition reads

$$F_{\text{Coulomb}}(A) = (\partial^\mu - n^\mu n^\nu \partial_\nu) A_\mu = \partial^i A_i \,,$$

where we have chosen $n^\mu = \delta_0^\mu$. The gauge defined by this gauge condition is called *Coulomb gauge*. It is not a Lorentz scalar, but an SO(3)-scalar. Note that formally, Coulomb gauge becomes Landau gauge in the limit $n^\mu \to 0$.

In this work, however, we shall only be concerned with covariant gauges.

We could now further simplify the ghost term. However, since in linear covariant gauges the ghosts decouple, we are not interested in the ghost term and from here on discard it altogether.[18]

With a linear covariant gauge fixing function, the gauge fixing part of the classical effective action is given by:

$$S_{\text{gf}}[A] = -\frac{1}{2\xi}\int_x [\partial^\mu A_\mu(x)]^2 = \frac{1}{2\xi}\int_x A_\mu(x)\,\partial^\mu\partial^\nu A_\nu(x) \tag{2.23}$$

where we have integrated by parts in the last step. It is also useful to integrate the photon part of the effective action by parts, yielding:

$$S_{\text{g}}[A] = -\frac{1}{4}\int_x F_{\mu\nu}(x)F^{\mu\nu}(x) = \frac{1}{2}\int_x A_\mu(x)(g^{\mu\nu}\Box - \partial^\mu\partial^\nu)A_\nu(x)\,. \tag{2.24}$$

The photon and the gauge fixing part of the effective action can then be combined to give:

$$S_{\text{g}}[A] + S_{\text{gf}}[A] = \frac{1}{2}\int_x A_\mu(x)[g^{\mu\nu}\Box - (1-\xi)\partial^\mu\partial^\nu]A_\nu(x)\,. \tag{2.25}$$

From now on, we will call the original classical action $S[A]$ together with the gauge fixing part $S_{\text{gf}}[A]$ the "classical effective action" $S_{\text{eff}}$, i. e.

$$\begin{aligned} S_{\text{eff}}[A,\overline{\psi},\psi] &= S[A,\overline{\psi},\psi] + S_{\text{gf}}[A] \\ &= \int_x \left\{ \frac{1}{2} A_\mu \Big[ g^{\mu\nu}\Box - (1-\xi)\,\partial^\mu\partial^\nu \Big] A_\nu + \overline{\psi}(\mathrm{i}\,\gamma^\mu \mathrm{D}_\mu - m)\psi \right\} . \end{aligned} \tag{2.26}$$

The generating functional of correlation functions is then given by:

$$Z[J,\eta,\overline{\eta}] = \int \mathcal{D}A\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi\, \exp\!\Big(\mathrm{i}\Big\{ S_{\text{eff}}[A,\overline{\psi},\psi] + \int_x (A_\mu J^\mu + \overline{\psi}\,\eta + \overline{\eta}\,\psi) \Big\}\Big)\,. \tag{2.27}$$

## 2.2 Nonequilibrium Quantum Field Theory

The next task is to formulate a quantum field theory out-of-equilibrium. Speaking of equilibrium or nonequilibrium implies a many-particle system, which has to be described by statistical means. The appropriate object to implement the state of such a system is the density operator. The question therefore is how to implement the density operator in the quantum field theory as defined by its path integral, which we will turn to next.

[18] One exception is the calculation of the energy-momentum tensor in App. B: Although the ghosts decouple, they do carry energy and momentum and therefore have to be included in its calculation.

### 2.2.1 The Density Operator

It is instructive to have a look at a *thermal* density operator first, i. e. a density operator which describes the statistics of a system which is in thermal equilibrium. We will find that the thermal equilibrium density operator is rather peculiar, which prohibits a simple generalization of the equilibrium case to the nonequilibrium case.

#### Thermal Equilibrium

The thermal density operator for a theory governed by a Hamiltonian $H$ at inverse temperature $\beta$ is given by

$$\rho = \frac{\mathrm{e}^{-\beta H}}{\mathrm{tr}(\mathrm{e}^{-\beta H})} = \mathcal{N}\,\mathrm{e}^{-\beta H}\,, \tag{2.28}$$

where the denominator is just a normalization constant ensuring that the trace of the density operator is unity and hence allows for a proper probabilistic interpretation.[19]

The peculiar property of the thermal density operator (2.28) is its similarity to the time evolution operator $U(t) = \mathrm{e}^{-\mathrm{i}Ht}$. In fact, if one formally admits complex times, we have

$$\rho = \mathcal{N}\,U(-\mathrm{i}\,\beta)\,, \tag{2.29}$$

i. e. the density operator can be formally interpreted as an operator evolving the system from the initial time (which can, without loss of generality, assumed to be zero) along the negative imaginary axis to the final "time" $-\mathrm{i}\,\beta$. The advantage of this formal analogy of the thermal density operator to the time evolution operator is that one essentially gets a path integral representation of the thermal density operator for free, since the path integral representation of the time evolution operator is well-known.[20]

It is also immediately clear now that there is no simple generalization of the thermal density operator to a general (nonequilibrium) one, since in general, there is no way of interpreting a given density operator as a time evolution operator. It is therefore necessary to follow a different approach as in thermal equilibrium.

#### General Density Operator

It is helpful to start with the definition of the generating functional as the expectation value of an expression involving quantum field operators. In vacuum (represented by the

[19] $\mathrm{tr}(\rho) = 1$ corresponds to the fact that probabilities need to sum up to unity. Further properties of the density operator are that it has to be hermitean, $\rho = \rho^\dagger$, corresponding to the fact that probabilities are real numbers, and that it has to be positive semidefinite, corresponding to the fact that probabilities have to be nonnegative. Altogether, these properties guarantee that probabilities lie in the interval $[0, 1]$.

[20] In fact, things are slightly more complicated. Due to the existence of the so-called KMS condition [Kub57, MS59] which relates the values of fields at $t = 0$ to their values at $t = -\mathrm{i}\,\beta$ (they are identical for bosons and negatives of each other for fermions), the integration in the path integral is restricted. The KMS condition is a boundary condition and can be interpreted as compactifying complex time on a circle of circumference $\beta$.

state vector $|0\rangle$), it is given by:

$$Z[J,\eta,\overline{\eta}] = \left\langle 0 \,\middle|\, \mathrm{T}\exp\Big(\mathrm{i}\int_x (\mathcal{A}_\mu J^\mu + \overline{\eta}\,\Psi + \overline{\Psi}\,\eta)\Big) \,\middle|\, 0 \right\rangle , \tag{2.30}$$

where $\mathcal{A}_\mu$ is the photon quantum field operator, $\Psi$ is the fermion quantum field operator, and T is the time-ordering symbol. More general states can be described by statistical means by introducing a density operator $\rho$. The generating functional then reads:

$$Z_\rho[J,\eta,\overline{\eta}] = \mathrm{Tr}\Big(\rho[\mathcal{A},\overline{\Psi},\Psi]\,\mathrm{T}\exp\Big(\mathrm{i}\int_x (\mathcal{A}_\mu J^\mu + \overline{\eta}\,\Psi + \overline{\Psi}\,\eta)\Big)\Big) , \tag{2.31}$$

and the density operator is normalized such that $\mathrm{Tr}(\rho) = 1$. Vacuum is then described by the density operator $\rho = |0\rangle\langle 0|$, i. e. by the projection operator onto the vacuum state.

Since the quantum field operators are Heisenberg operators, they depend on time. Let us assume that at some initial time $t_0$, we have:

$$\begin{aligned}
\mathcal{A}_\mu(t_0,\boldsymbol{x})\Big|A,\psi,\overline{\psi};t_0\Big\rangle &= A_\mu(t_0,\boldsymbol{x})\Big|A,\psi,\overline{\psi};t_0\Big\rangle , \\
\overline{\Psi}(t_0,\boldsymbol{x})\Big|A,\psi,\overline{\psi};t_0\Big\rangle &= \overline{\psi}(t_0,\boldsymbol{x})\Big|A,\psi,\overline{\psi};t_0\Big\rangle , \\
\Psi(t_0,\boldsymbol{x})\Big|A,\psi,\overline{\psi};t_0\Big\rangle &= \psi(t_0,\boldsymbol{x})\Big|A,\psi,\overline{\psi};t_0\Big\rangle ,
\end{aligned} \tag{2.32}$$

which defines eigenvectors of the Heisenberg quantum field operators. With respect to the basis of eigenvectors, the trace of the generating functional can then be written as:

$$\begin{aligned}
&Z_\rho[J,\eta,\overline{\eta}] \\
&\quad = \int \mathcal{D}A^{(1)}\,\mathcal{D}\overline{\psi}^{(1)}\,\mathcal{D}\psi^{(1)}\,\mathcal{D}A^{(2)}\,\mathcal{D}\overline{\psi}^{(2)}\,\mathcal{D}\psi^{(2)} \\
&\qquad \cdot \Big\langle A^{(1)},\overline{\psi}^{(1)},\psi^{(1)};\tau(0)\Big|\rho[\mathcal{A},\overline{\Psi},\Psi]\Big|A^{(2)},\overline{\psi}^{(2)},\psi^{(2)};\tau(1)\Big\rangle \\
&\qquad \cdot \left\langle A^{(2)},\overline{\psi}^{(2)},\psi^{(2)};\tau(1) \,\middle|\, \mathrm{T}\exp\Big(\mathrm{i}\int_x \Big(\mathcal{A}_\mu J^\mu + \overline{\eta}\,\Psi + \overline{\Psi}\,\eta\Big)\Big) \,\middle|\, A^{(1)},\overline{\psi}^{(1)},\psi^{(1)};\tau(0) \right\rangle ,
\end{aligned} \tag{2.33}$$

where

$$\begin{aligned}
\mathcal{D}A^{(i)} &= \prod_{\mu=0}^{3}\prod_{\boldsymbol{x}} \mathrm{d}A_\mu^{(i)}(\boldsymbol{x}) = \prod_{\boldsymbol{x}} \mathrm{d}A_0^{(i)}(\boldsymbol{x})\ \mathrm{d}A_1^{(i)}(\boldsymbol{x})\ \mathrm{d}A_2^{(i)}(\boldsymbol{x})\ \mathrm{d}A_3^{(i)}(\boldsymbol{x}) , \\
\mathcal{D}\overline{\psi}^{(i)} &= \prod_{\boldsymbol{x}} \mathrm{d}\overline{\psi}^{(i)}(\boldsymbol{x}) , \\
\mathcal{D}\psi^{(i)} &= \prod_{\boldsymbol{x}} \mathrm{d}\psi^{(i)}(\boldsymbol{x}) .
\end{aligned}$$

Note that the path integral does not extend over all field configurations, but only over those at time $t_0$, since the density matrix is only given at $t_0$. Of course, since we are interested in studying time evolution, it is not meaningful to set up a generating functional which only yields correlation functions at a single time $t_0$. Instead of considering a time path which extends infinitely into past and future (as one usually does when considering the vacuum and is interested in asymptotic states) or a fixed time only, we will therefore consider a *closed time path* (CTP)[21] which starts at the initial time $t_0$, extends to some

[21] Also called *Schwinger–Keldysh contour* [Sch61, Kel64].

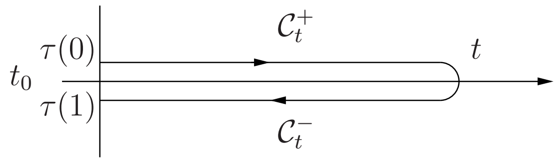


Figure 2.1: Sketch of the Ctp. Note that the displacement away from the real time axis of the two branches is only to distinguish them pictorially; the domain of the Ctp is the real interval $[t_0; t] \subset \mathbb{R}$.

finite time $t$ and then returns to $t_0$, see Fig. (2.1). The Ctp can be parametrized as follows. We consider the forward branch $\mathcal{C}_t^+$ and the backward branch $\mathcal{C}_t^-$ separately:

$$\begin{aligned}\mathcal{C}_t^+ : [0,1] &\to [t_0, t]\,,\\ \lambda &\mapsto \tau^+(\lambda) = \lambda\, t\,,\\ \mathcal{C}_t^- : [0,1] &\to [t_0, t]\,,\\ \lambda &\mapsto \tau^-(\lambda) = (1-\lambda)\, t\,,\end{aligned}$$

and then compose the whole Ctp from the two branches according to:

$$\begin{aligned}\mathcal{C}_t = \mathcal{C}_t^+ \oplus \mathcal{C}_t^- : [0,1] &\to [t_0, t]\,,\\ \lambda \mapsto \tau(\lambda) &= \begin{cases}\tau^+(2\lambda)\,; & 0 \le \lambda \le \frac{1}{2}\,,\\ \tau^-(2\lambda-1)\,; & \frac{1}{2} \le \lambda \le 1\end{cases}\\ &= \begin{cases}2\lambda\, t\,; & 0 \le \lambda \le \frac{1}{2}\,,\\ 2(1-\lambda)t\,; & \frac{1}{2} \le \lambda \le 1\,.\end{cases}\end{aligned}$$

The Ctp is a loop since the initial and terminal points coincide, $\tau(0) = \tau(1) = t_0$. The maximum time is given by $\tau(1/2) = t$. Note that instead of time ordering, we now have a path ordering: $\tau(\lambda_1)$ is later than $\tau(\lambda_2)$ if $\lambda_1 > \lambda_2$. This implies in particular that every time on $\mathcal{C}^-$ is later than every time on $\mathcal{C}^+$.

This enables us to evaluate the density matrix at the fixed time $t_0$ only and nevertheless obtain correlation functions at later times. In the following, we will therefore assume all (space)time integrals to be defined on the Ctp.

In order to turn the above expression into a proper path integral, we need the path integral representation for Eq. (2.33). According to what we have said in the preceding paragraph, it is clear that it is given by:

$$\begin{aligned}&\left\langle A^{(2)}, \overline{\psi}^{(2)}, \psi^{(2)} \,\middle|\, \mathrm{T}\exp\!\left(\mathrm{i}\int_{\mathcal{C}_t,\boldsymbol{x}} (\mathcal{A}_\mu J^\mu + \overline{\eta}\,\Psi + \overline{\Psi}\,\eta)\right) \middle|\, A^{(1)}, \overline{\psi}^{(1)}, \psi^{(1)} \right\rangle\\ &= \int\limits_{\substack{A(\tau(0),\boldsymbol{x})=A^{(1)}(\boldsymbol{x}),A(\tau(1),\boldsymbol{x})=A^{(2)}(\boldsymbol{x})\\ \overline{\psi}(\tau(0),\boldsymbol{x})=\overline{\psi}^{(1)}(\boldsymbol{x}),\overline{\psi}(\tau(1),\boldsymbol{x})=\overline{\psi}^{(2)}(\boldsymbol{x})\\ \psi(\tau(0),\boldsymbol{x})=\psi^{(1)}(\boldsymbol{x}),\psi(\tau(1),\boldsymbol{x})=\psi^{(2)}(\boldsymbol{x})}} \mathcal{D}A\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi\, \exp\!\left(\mathrm{i}\left\{S_{\mathrm{eff}}[A, \overline{\psi}, \psi] + \int_{\mathcal{C}_{x^0},\boldsymbol{x}} [A_\mu J^\mu + \overline{\eta}\,\psi + \overline{\psi}\,\eta]\right\}\right)\end{aligned}$$

with the integral over the CTP

$$\int_{\mathcal{C}_{x^0},\boldsymbol{x}} = \int_{\mathcal{C}_{x^0}} \mathrm{d}x^0 \int \mathrm{d}^3x \,.$$

Then:

$$\begin{aligned}
&\Big\langle A^{(1)}, \overline{\psi}^{(1)}, \psi^{(1)}; \tau(0) \Big| \rho[\mathcal{A}, \overline{\Psi}, \Psi] \Big| A^{(2)}, \overline{\psi}^{(2)}, \psi^{(2)}; \tau(1) \Big\rangle \\
&\quad = \rho[A^{(1)}, \overline{\psi}^{(1)}, \psi^{(1)}, A^{(2)}, \overline{\psi}^{(2)}, \psi^{(2)}] \\
&\quad = \mathcal{N} \exp\Big(\mathrm{i}\, \tilde{f}\Big[A^{(1)}, \overline{\psi}^{(1)}, \psi^{(1)}, A^{(2)}, \overline{\psi}^{(2)}, \psi^{(2)}\Big]\Big) \,.
\end{aligned}$$

Note that there is no assumption involved in the last line; it is just a parametrization of the density matrix resembling a polar decomposition; see also Ref. [CHK+94]. If this were a proper polar decomposition, $\mathcal{N}$ and $\tilde{f}$ would have to be real, and then $\rho$ were complex in general. Since $\rho$ is real, however, $\tilde{f}$ must in fact be imaginary. Since the exponential is real, $\mathcal{N}$ is then really just a constant (i. e. independent of the fields).

It is then useful to do a Taylor expansion of $\tilde{f}$:

$$\begin{aligned}
&\tilde{f}\Big[A^{(1)}, \overline{\psi}^{(1)}, \psi^{(1)}, A^{(2)}, \overline{\psi}^{(2)}, \psi^{(2)}\Big] \\
&\quad = \alpha_{(0,0,0)} \\
&\qquad + \int_{\boldsymbol{x}} \Big[ \alpha^{(1)\mu}_{(1,0,0)}(\boldsymbol{x}) A^{(1)}_\mu(\boldsymbol{x}) + \alpha^{(2)\mu}_{(1,0,0)}(\boldsymbol{x}) A^{(2)}_\mu(\boldsymbol{x}) \\
&\qquad\qquad + \alpha^{(1)}_{(0,1,0)}(\boldsymbol{x}) \overline{\psi}^{(1)}(\boldsymbol{x}) + \alpha^{(2)}_{(0,1,0)}(\boldsymbol{x}) \overline{\psi}^{(2)}(\boldsymbol{x}) \\
&\qquad\qquad + \alpha^{(1)}_{(0,0,1)}(\boldsymbol{x}) \psi^{(1)}(\boldsymbol{x}) + \alpha^{(2)}_{(0,0,1)}(\boldsymbol{x}) \psi^{(2)}(\boldsymbol{x}) \\
&\qquad + \int_{\boldsymbol{x},\boldsymbol{y}} \Big[ \alpha^{(1,1)\mu\nu}_{(2,0,0)}(\boldsymbol{x},\boldsymbol{y}) A^{(1)}_\mu(\boldsymbol{x}) A^{(1)}_\nu(\boldsymbol{y}) + \alpha^{(1,2)\mu\nu}_{(2,0,0)}(\boldsymbol{x},\boldsymbol{y}) A^{(1)}_\mu(\boldsymbol{x}) A^{(2)}_\nu(\boldsymbol{y}) \\
&\qquad\qquad + \alpha^{(2,1)\mu\nu}_{(2,0,0)}(\boldsymbol{x},\boldsymbol{y}) A^{(2)}_\mu(\boldsymbol{x}) A^{(1)}_\nu(\boldsymbol{y}) + \alpha^{(2,2)\mu\nu}_{(2,0,0)}(\boldsymbol{x},\boldsymbol{y}) A^{(2)}_\mu(\boldsymbol{x}) A^{(2)}_\nu(\boldsymbol{y}) \\
&\qquad\qquad + \alpha^{(1,1)}_{(0,1,1)}(\boldsymbol{x},\boldsymbol{y}) \overline{\psi}^{(1)}(\boldsymbol{x}) \psi^{(1)}(\boldsymbol{y}) + \alpha^{(1,2)}_{(0,1,1)}(\boldsymbol{x},\boldsymbol{y}) \overline{\psi}^{(1)}(\boldsymbol{x}) \psi^{(2)}(\boldsymbol{y}) \\
&\qquad\qquad + \alpha^{(2,1)}_{(0,1,1)}(\boldsymbol{x},\boldsymbol{y}) \overline{\psi}^{(2)}(\boldsymbol{x}) \psi^{(1)}(\boldsymbol{y}) + \alpha^{(2,2)}_{(0,1,1)}(\boldsymbol{x},\boldsymbol{y}) \overline{\psi}^{(2)}(\boldsymbol{x}) \psi^{(2)}(\boldsymbol{y}) \Big] \\
&\qquad + \ldots \\
&\quad = \alpha_{(0,0,0)} \\
&\qquad + \sum_{a=1}^{2} \int_{\boldsymbol{x}} \Big[ \alpha^{(a)\mu}_{(1,0,0)}(\boldsymbol{x}) A^{(a)}_\mu(\boldsymbol{x}) + \alpha^{(a)\mu}_{(0,1,0)}(\boldsymbol{x}) \overline{\psi}^{(a)}(\boldsymbol{x}) + \alpha^{(a)\mu}_{(0,0,1)}(\boldsymbol{x}) \psi^{(a)}(\boldsymbol{x}) \Big] \\
&\qquad + \sum_{a,b=1}^{2} \int_{\boldsymbol{x},\boldsymbol{y}} \Big[ \alpha^{(a,b)\mu\nu}_{(2,0,0)}(\boldsymbol{x},\boldsymbol{y}) A^{(a)}_\mu(\boldsymbol{x}) A^{(b)}_\nu(\boldsymbol{y}) + \alpha^{(a,b)}_{(0,1,1)}(\boldsymbol{x},\boldsymbol{y}) \overline{\psi}^{(a,b)}(\boldsymbol{x}) \psi^{(b)}(\boldsymbol{y}) \Big] \\
&\qquad + \ldots \,.
\end{aligned}$$

By introducing delta distributions which make the quantities nonvanishing only at the initial time $t_0$ (corresponding to the endpoints $\tau(0)$ and $\tau(1)$ of the CTP), the functional can be rewritten as:

$$
\begin{aligned}
f[A,\overline{\psi},\psi] = \alpha_{(0,0,0)} & \\
&+ \int_{\mathcal{C}_{x^0},\boldsymbol{x}} \Big[\alpha^{\mu}_{(1,0,0)}(x) A_\mu(x) + \alpha_{(0,1,0)}(x)\overline{\psi}(x) + \alpha_{(0,0,1)}(x)\psi(x)\Big] \\
&+ \frac{1}{2}\int_{\mathcal{C}_{x^0},\boldsymbol{x},\mathcal{C}_{y^0},\boldsymbol{y}} \Big[\alpha^{\mu\nu}_{(2,0,0)}(x,y) A_\mu(x) A_\nu(y) + \alpha_{(0,1,1)}(x,y)\overline{\psi}(x)\psi(y) + \ldots\Big] \\
&+ \ldots .
\end{aligned}
$$

Note that this is just a Taylor expansion of $f$ with Taylor coefficients

$$
\begin{aligned}
\alpha_{(0,0,0)} &= f[0,0,0]\,, \\
\alpha^{\mu}_{(1,0,0)}(x) &= \left.\frac{\delta f[A,\overline{\psi},\psi]}{\delta A_\mu(x)}\right|_{A=0,\overline{\psi}=0,\psi=0} \\
\alpha_{(0,1,0)}(x) &= \left.\frac{\delta f[A,\overline{\psi},\psi]}{\delta \overline{\psi}(x)}\right|_{A=0,\overline{\psi}=0,\psi=0} \\
\alpha_{(0,0,1)}(x) &= \left.\frac{\delta f[A,\overline{\psi},\psi]}{\delta \psi(x)}\right|_{A=0,\overline{\psi}=0,\psi=0} \\
\alpha^{\mu\nu}_{(2,0,0)}(x,y) &= \left.\frac{\delta^2 f[A,\overline{\psi},\psi]}{\delta A_\mu(x)\delta A_\nu(y)}\right|_{A=0,\overline{\psi}=0,\psi=0} \\
\alpha_{(0,1,1)}(x,y) &= \left.\frac{\delta^2 f[A,\overline{\psi},\psi]}{\delta \overline{\psi}(x)\delta\psi(y)}\right|_{A=0,\overline{\psi}=0,\psi=0}\,,
\end{aligned}
$$

where the index $(i,j,k)$ denotes the number of variations with respect to $A$, $\overline{\psi}$ and $\psi$, respectively. Note that in the above expression, already at second order we have left out terms formally appearing in the Taylor expansion which, however, vanish identically.[22] In fact, the $n$th order of the Taylor expansion contains all $\alpha_{(i,j,k)}$ with $i+j+k=n$.

The generating functional of correlation functions then reads:

$$
\begin{aligned}
&Z_\rho[J,\eta,\overline{\eta}] \\
&\quad= \int \mathcal{D}A^{(1)}\,\mathcal{D}\overline{\psi}^{(1)}\,\mathcal{D}\psi^{(1)}\,\mathcal{D}A^{(2)}\,\mathcal{D}\overline{\psi}^{(2)}\,\mathcal{D}\psi^{(2)}\rho\Big[A^{(1)},\overline{\psi}^{(1)},\psi^{(1)},A^{(2)},\overline{\psi}^{(2)},\psi^{(2)}\Big] \\
&\qquad\qquad \int\limits_{\substack{A(\tau(0),\boldsymbol{x})=A^{(1)}(\boldsymbol{x}),A(\tau(1),\boldsymbol{x})=A^{(2)}(\boldsymbol{x})\\ \overline{\psi}(\tau(0),\boldsymbol{x})=\overline{\psi}^{(1)}(\boldsymbol{x}),\overline{\psi}(\tau(1),\boldsymbol{x})=\overline{\psi}^{(2)}(\boldsymbol{x})\\ \psi(\tau(0),\boldsymbol{x})=\psi^{(1)}(\boldsymbol{x}),\psi(\tau(1),\boldsymbol{x})=\psi^{(2)}(\boldsymbol{x})}} \mathcal{D}A\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi \exp\Big(\mathrm{i}\Big\{S_{\text{eff}}[A,\overline{\psi},\psi] + \int_x \Big(A_\mu J^\mu + \overline{\eta}\,\psi + \overline{\psi}\,\eta\Big)\Big\}\Big) \\
&\quad= \int \mathcal{D}A\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi \exp\Big(\mathrm{i}\Big\{S_{\text{eff}}[A,\overline{\psi},\psi] + \int_x \Big(A_\mu J^\mu + \overline{\eta}\,\psi + \overline{\psi}\,\eta\Big) + f[A,\overline{\psi},\psi]\Big\}\Big)\,.
\end{aligned}
\tag{2.34}
$$

[22]For instance, at second order, the terms left out correspond to mixing terms involving, e. g., a photon field and a fermion field, or two fermion fields. They correspond to correlators which vanish at initial time.

This is the generating functional for correlation functions implementing a general density operator $\rho$. The only differences to the vacuum generating functional are the integration over the CTP and the additional term in the exponential stemming from the parametrization of the density matrix. Note that formally, this term adds new interactions of arbitrary order to the theory, which, however, only act at the initial time $t_0$ and therefore only influence the initial state of the system. Note that the 0th order term of the Taylor expansion of $f$ only generates an overall factor $\mathcal{N} \exp(\mathrm{i}\, \alpha_{(0,0,0)})$ which we will ignore in the following, i. e. we can without loss of generality assume that $\alpha_{(0,0,0)} = 0$.[23] Further note that if $f$ were linear in the fields, the coefficients of its Taylor expansion could be absorbed into the external sources to make the generating functional look even more similar to the vacuum case. However, a linear $f$ corresponds to a rather trivial density operator which is of no interest for us. We will come back to this issue after the next subsection.

Finally note that, since so far we have not restricted the density operator at all, it should in particular include the *thermal* density operator (2.28) considered earlier. For an interacting theory, i. e. for a theory with a Hamiltonian which includes powers of quantum field operators higher than two, however, the corresponding density matrix is very complicated in the sense that the Taylor expansion of $f$ will in general not terminate at finite order, leading to arbitrarily complicated interaction terms at initial time.

Only for a free ("Gaussian") theory, i. e. for a theory with quadratic Hamiltonian, $f$ becomes simple, i. e. its Taylor series terminates at second order. This corresponds to a *Gaussian* density matrix. Restricting to Gaussian density matrices in the thermal case therefore implies restricting to free theories only. For general density matrices, however, this is not true: Gaussian density matrices do not necessarily imply considering free theories only. Considering Gaussian density matrices therefore forces one to consider nonequilibrium systems if one is interested in interacting theories. The importance of Gaussian density matrices (besides the fact that they are simple) will become clear shortly.

### 2.2.2 Secularity and Resummation

Nonequilibrium QFT in a perturbative approach is doomed for failure. In order to see why and to be concrete, let us assume that we wanted to study the thermalization of some quantum field from first principles and think about which approach to use. The logical first guess is perturbation theory, i. e. an expansion in the free propagators of the theory. This usually works very well if one intends to study e. g. scattering processes in vacuum if the coupling is weak, like collisions in a particle collider. The reason is that the interaction is highly localized in spacetime: One starts with an asymptotic state in the infinite past where nothing is interacting, then "turns on" the interaction corresponding to the collision of the particles, and then looks at an asymptotic state in the infinite future where again nothing is interacting.

Another example are bound states (still in vacuum). Here, things are very different from the particle collisions considered in the previous paragraph: Although the interaction

[23] To be more precise, it can be used to ensure that the trace of the density operator is unity.

is still localized in space (at least for short-range interactions), the interaction in time is now continuous. In a perturbative calculation, this corresponds to the fact that a large number of diagrams contributes (possibly infinitely many for a stable bound state). The study of bound states is therefore clearly not practicable in perturbation theory.

Another situation which is somewhat more close to our concern is thermal field theory, i. e. the statistical theory which is obtained when the density operator is a thermal one. Let us for the sake of concreteness consider a thermal $\lambda\,\phi^4$ theory ($\lambda \ll 1$) for the case that the temperature $T$ of the background is very large compared to the mass $m$ of the theory (so that we can assume it to be massless). Consider the one-loop self-energy $\Sigma_{\text{1-loop}}$, which is a tadpole and hence local. Therefore, its Fourier transform does not depend on momentum and is just a number. For dimensional reasons,

$$\Sigma_{\text{1-loop}}(G_0) \sim \lambda \int_p G_0(p) \sim \lambda\, T^2 \sim m_{\text{th}}^2\,,$$

where $G_0$ is the free thermal propagator[24], and we have introduced the *thermal mass* $m_{\text{th}}$. Obviously, the perturbative one-loop photon self-energy is, up to a numerical factor, just the thermal mass squared and hence constant. From the Dyson–Schwinger equation (DSE) [Dys49, Sch51a, Sch51b], relating the full propagator to the free propagator and the self-energy, it follows that the one-loop propagator is determined by

$$G^{-1}(p) = G_0^{-1}(p) - \Sigma_{\text{1-loop}}(G_0) = G_0^{-1}(p) - m_{\text{th}}^2 \approx p_0^2 - (\boldsymbol{p}^2 + \lambda\, T^2)\,,$$

so that it is approximately given by

$$G(p_0, \boldsymbol{p}) = \frac{1}{p_0^2 - (\boldsymbol{p}^2 + m_{\text{th}}^2)}\,.$$

The important point now is to note that there are two different scales which have to be distinguished: The scale set by the temperature, and the scale set by the product of the square root of the coupling constant times the temperature.[25] For "hard" spatial momenta of the order of the temperature, $|\boldsymbol{p}| \sim T \sim m_{\text{th}}/\sqrt{\lambda}$, the effect of the thermal mass leads to a small correction of the propagator compared to the free one since $\lambda \ll 1$ by definition. For soft momenta $|\boldsymbol{p}| \sim \sqrt{\lambda}\,T \sim m_{\text{th}}$, however, the correction is of the same order as the inverse propagator. Therefore, perturbation theory breaks down at the soft scale, no matter how small the coupling $\lambda$.

The physical reason is again the continuous interaction of particles with the thermal bath. If the momentum of some particle is hard, it does not "see" the thermal bath, but if it is soft, it gets influenced severely by its propagation through the bath.

---

[24] The thermal propagator in real time can be split into a vacuum and a thermal part [LB00]. Here, we are only interested in the thermal part, which is finite (while the vacuum part is UV divergent and would require regularization), and discard the vacuum part altogether.

[25] Remember that we have assumed that the mass is negligible compared to the temperature, so that the mass does not constitute a scale.

The solution to the problem is to do perturbation theory not with the free propagator, but with a propagator which already includes thermal effects like, for instance, the one-loop propagator considered above. The DSE then becomes

$$G^{-1}(p) = G_0^{-1}(p) - \Sigma_{\text{1-loop}}(G)\,,$$

i.e. the self-energy is calculated not with the free propagator, but self-consistently with the propagator we are solving for. The full propagator itself is then recursively given by

$$G(p) = G_0(p) - G_0(p)\Sigma_{\text{1-loop}}(G)G(p)\,,$$

which corresponds to the resummation of an infinite number of perturbative diagrams.

As we have seen, there are situations where naive perturbation theory does not work in thermal equilibrium, and some kind of resummation has to be introduced. Similarly, it turns out that out-of-equilibrium, perturbation theory fails as well. The reason for the failure is again easy to understand and can be illustrated by the example of a simple anharmonic oscillator. Consider the differential equation [Ber05]

$$\ddot{x}(t) + x(t) = -\varepsilon\,\dot{x}(t) - \frac{\varepsilon^3 x^3(t)}{1 - \varepsilon^2 x^2(t)} \tag{2.35}$$

which depends on a small parameter $\varepsilon \ll 1$. For $\varepsilon = 0$, this differential equation obviously reduces to that of a simple harmonic oscillator, while for $\varepsilon > 0$, we have a friction term (i.e. a term depending on $\dot{x}(t)$—the first term on the right-hand side) as well as anharmonic terms (i.e. terms depending on powers greater than one of $x(t)$—the second one on the right-hand side). In order to be able to assess the importance of the different terms, it is convenient to expand the fraction on the right-hand side into a Taylor series in $\varepsilon$. One obtains [Ber05]:

$$\ddot{x}(t) + x(t) = -\varepsilon\,\dot{x}(t) - \sum_{n=3}^{\infty} \varepsilon^n x^n(t) = -\varepsilon\,\dot{x}(t) - \varepsilon^3 x^3(t) - \varepsilon^5 x^5(t) - \dots\,.$$

**Perturbative Approach**

A perturbative approach would now proceed to expand the solution $x(t)$ in a series in the expansion parameter $\varepsilon$ and plug it back into the differential equation. By comparing powers in $\varepsilon$, one then obtains an infinite hierarchy of equations which can be solved iteratively. With

$$x(t) = x_0(t) + \varepsilon\,x_1(t) + \frac{1}{2}\varepsilon^2 x_2^2(t) + \dots\,, \tag{2.36}$$

one obtains:

$$\ddot{x}_0(t) + x_0(t) = 0\,, \tag{2.37a}$$

$$\ddot{x}_1(t) + x_1(t) = -\varepsilon\,\dot{x}_0(t)\,, \tag{2.37b}$$

$$\vdots$$

where the subscript refers to the power in $\varepsilon$. In order to solve these equations, we need to specify initial conditions. With $x(0) = 1$ and $\dot{x}(0) = -\varepsilon/2$, one can easily solve the first equation:

$$x_0(t) = \cos(t)\,.$$

The second equation then reads:

$$\ddot{x}_1(t) + x_1(t) = \sin(t)\,.$$

It is immediately clear that this equation is problematic: It is the equation of a driven harmonic oscillator, where the driving term $\sin(t)$ has the same frequency as the eigenfrequency of the harmonic oscillator, namely unity in this example. We therefore have a resonance, and we expect the solution to grow without bounds, i. e. to diverge in time. In fact, it can easily be checked that the solution is given by:

$$x_1(t) = -\frac{1}{2}\,t\cos(t)\,,$$

so it indeed diverges in time, i. e. is secular.

The full solution then reads:

$$x(t) = x_0(t) + \varepsilon\,x_1(t) + \mathcal{O}(\varepsilon^2) = \cos(t) - \frac{1}{2}\,\varepsilon\,t\cos(t) + \mathcal{O}(\varepsilon^2)\,. \tag{2.38}$$

It is obvious that the perturbative solution does not give a uniform approximation to the solution since the quality of the solution does not depend on $\varepsilon$ alone, but in fact on the product $\varepsilon\,t$. It is then clear that the approximation is only good as long as $\varepsilon\,t \ll 1$, i. e. for times $t \ll 1/\varepsilon$. The conclusion is that for a perturbative solution, there is *always* some time $\bar{t} = 1/\varepsilon$ where the solution breaks down. A perturbative approach is therefore not suitable for studying time evolution problems in nonequilibrium situations.

### Resummed approach

There are different methods to solve the problem of secularity by resumming the secular terms appearing at each perturbative order (except for the zeroth order) into finite terms which do not diverge in time.[26] The method based on the 2PI effective action employed in this work proceeds as follows: Instead of expanding not only the differential equation (2.35) in the parameter $\varepsilon$, but the solution (2.36) as well, we *only* expand the differential equation and truncate it at some order in $\varepsilon$. This means that we do not have to solve a series of differential equations, but *exactly one* equation. For instance, at zero order in $\varepsilon$, the equation reads

$$\ddot{x}(t) + x(t) = 0\,,$$

which of course coincides with the corresponding equation in the perturbative approach. To first order in $\varepsilon$, the equation to solve is given by

$$\ddot{x}(t) + x(t) = -\varepsilon\,\dot{x}(t)\,.$$

[26] Like the *dynamical renormalization group*, see e. g. Ref. [BdV03].

Note that in contrast to the first-order perturbative equation (2.37b), this equation depends only on $x(t)$ itself, not on some "external" source coming from a lower-order solution—it is a self-consistent equation.

The solution then reads

$$x(t) = \cos\left(\sqrt{1 - \frac{1}{4}\,\varepsilon^2}\; t\right) \mathrm{e}^{-\frac{1}{2}\varepsilon\, t} \,. \tag{2.39}$$

The first thing to note is that the solution looks qualitatively right, i. e. it shows damping and in particular does not exhibit secular terms. Further, the frequency becomes "renormalized", i. e. shifted from 1 to $\sqrt{1-\varepsilon^2/4}$.

The reason that this solution does not contain secular terms is that they have been resummed into the cosine and the exponential. This can be seen by expanding the solution perturbatively in $\varepsilon$ again:

$$x(t) = \cos(t) - \frac{1}{2}\,\varepsilon\, t\cos(t) + \mathcal{O}(\varepsilon^2)\,,$$

which just corresponds to the perturbative solution Eq. (2.38) of the corresponding order. From this point of view, the reason for the appearance of secular terms in the perturbative approach is therefore simply the fact that finite expansions of bounded functions need not be bounded (and in fact are in general not bounded).

The conclusion then is that solving an equation and expanding its solution do not commute.

### 2.2.3 The 2PI Effective Action

Now that it should be clear that perturbation theory will most likely not work out-of-equilibrium, we have to face the question which method to use in order to tackle nonequilibrium QFT. According to what has been said above, such a method will certainly have to implement some kind of resummation to accommodate for the continuous interaction with the system.

An approach which has proved to work very well for many nonequilibrium related problems is based on the 2PI effective action. Employing the 2PI effective action, it has been possible to numerically demonstrate the thermalization of scalar as well as fermionic theories from first principles, i. e. by directly solving equations of motion (EOMs) for the fundamental objects of a QFT, namely its correlation functions. The 2PI effective action has two very important features which are necessary to address nonequilibrium problems: It implements a highly efficient resummation, and it guarantees the conservation of global charges as required by the continuous symmetries of the given theory. In particular, the conservation of the total energy is crucial for problems regarding thermalization. A nice overview of applications of 2PI effective action techniques as well as other methods for dealing with nonequilibrium problems can be found in Ref. [Ber05] as well as in the monograph [CH08].

In order to understand the mode of action of the 2PI effective action, it is useful to briefly recapitulate the closely related 1PI effective action, which should be more familiar. The 1PI effective action is in turn related to the generating functional of correlation functions considered before. Starting from the generating functional for correlation functions, one first constructs the *generating functional of* connected *correlation functions* $W[J,\eta,\overline{\eta}]$, defined by

$$Z[J,\eta,\overline{\eta}] = \exp(\mathrm{i}\, W[J,\eta,\overline{\eta}])\,. \tag{2.40}$$

Correlation functions can be obtained from $W$ in the same way as from $Z$, i. e. by calculating variations with respect to the external sources. Correlation functions obtained from $W$, however, can be shown to always be connected, i. e. in a diagrammatic representation, the corresponding diagrams do not consist of separate graphs. $W$ therefore implements a more efficient way of storing the information contained in a QFT.

The 1PI effective action $\Gamma_{\mathrm{1PI}}$ can now be obtained from $W$ by doing a Legendre transform with respect to the sources:

$$\begin{aligned}\Gamma_{\mathrm{1PI}}[A,\overline{\psi},\psi] = W[&J(A,\overline{\psi},\psi),\eta(A,\overline{\psi},\psi),\overline{\eta}(A,\overline{\psi},\psi)]\\ &- \int_x \Big[A_\mu J^\mu(A,\overline{\psi},\psi) + \overline{\eta}(A,\overline{\psi},\psi)\,\psi + \overline{\psi}\,\eta(A,\overline{\psi},\psi)\Big]\,. \end{aligned}\tag{2.41}$$

Note that the dependence on the external sources has been traded for a dependence on the expectation values of the fields. Correlation functions can then be obtained by calculating variations of the 1PI effective action with respect to the expectation values of the fields, and the correlation functions thus obtained can be shown to be *one-particle irreducible*, i. e. in a diagrammatic representation, diagrams do not become disconnected by cutting one propagator line. They are also called *proper vertex functions*, and the 1PI effective action is correspondingly also called *generating functional of proper vertex functions*. Since every diagram contained in $W$ can be obtained by joining two diagrams contained in $\Gamma_{\mathrm{1PI}}$ by a propagator line, $\Gamma_{\mathrm{1PI}}$ implements an even more efficient way to store the information contained in a QFT. In a sense, the correlation functions obtained from $\Gamma_{\mathrm{1PI}}$ provide the most basic building blocks (propagators and proper vertices) from which diagrams can be constructed.

Since the 1PI effective action is a functional of the field expectation values, EOMs for the field expectation values can be obtained by a variational principle. Just like for the classical action, the EOMs for the field expectation values follow from requiring the variation of the 1PI effective action to vanish. Higher-order correlation functions can then be obtained by calculating variations of the 1PI effective action with respect to the fields and evaluating them at the solutions to the EOMs. For instance,

$$\left.\frac{\delta\Gamma_{\mathrm{1PI}}[A,\overline{\psi},\psi]}{\delta A_\mu}\right|_{\mathrm{phys}} = 0\,, \quad \left.\frac{\delta\Gamma_{\mathrm{1PI}}[A,\overline{\psi},\psi]}{\delta\overline{\psi}}\right|_{\mathrm{phys}} = 0\,, \quad \left.\frac{\delta\Gamma_{\mathrm{1PI}}[A,\overline{\psi},\psi]}{\delta\psi}\right|_{\mathrm{phys}} = 0\,, \tag{2.42}$$

where $\ldots|_{\mathrm{phys}}$ means that the values of the arguments of the 2PI effective action are such that they satisfy the equations. Then the inverse photon propagator, for instance, is given by:

$$\mathrm{i}\left(D^{-1}\right)^{\mu\nu}(x,y) = \left.\frac{\delta^2\Gamma_{1\mathrm{PI}}[A,\overline{\psi},\psi]}{\delta A_\mu(x)\delta A_\nu(y)}\right|_{\mathrm{phys}} ,$$

while the proper three-point function, i. e. the full electron-photon vertex, is given by:

$$\mathrm{i}\,\Gamma^\mu(x,y,z) = \left.\frac{\delta^3\Gamma_{1\mathrm{PI}}[A,\overline{\psi},\psi]}{\delta A_\mu(x)\delta\overline{\psi}(y)\delta\psi(z)}\right|_{\mathrm{phys}} ,$$

and so on for higher correlation functions. The important point to note here is that in order to obtain higher order than one-point functions, one first has to solve the EOMs (2.42) for the field expectation values.

The 2PI effective action is obtained in a very similar way as the 1PI effective action. However, instead of starting with a generating functional of correlation functions which depends on one external source for each field coupled linearly to the field, one introduces in addition external sources coupled *bilinearly* to the fields:[27]

$$\begin{aligned}
&Z_\rho[J,\eta,\overline{\eta},R,K] \\
&\quad= \int \mathcal{D}A\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi\,\exp\Bigg(\mathrm{i}\Bigg\{S_{\mathrm{eff}}[A,\overline{\psi},\psi] + f[A,\overline{\psi},\psi] \\
&\qquad\qquad + \int_x \Big[A_\mu(x)J^\mu(x) + \overline{\eta}(x)\,\psi(x) + \overline{\psi}(x)\,\eta(x)\Big] \\
&\qquad\qquad + \frac{1}{2}\int_{x,y}\Big[A_\mu(x)R^{\mu\nu}(x,y)A_\nu(y) + \overline{\psi}(x)K(x,y)\psi(y)\Big]\Bigg\}\Bigg) \\
&\quad= \int \mathcal{D}A\,\mathcal{D}\overline{\psi}\,\mathcal{D}\psi\,\exp\Bigg(\mathrm{i}\Bigg\{S_{\mathrm{eff}}[A,\overline{\psi},\psi] \\
&\qquad\qquad + \int_x \Big\{A_\mu(x)\Big[J^\mu(x) + \alpha^\mu_{(1,0,0)}(x)\Big] \\
&\qquad\qquad\quad + \Big[\overline{\eta}(x) + \alpha_{(0,1,0)}(x)\Big]\,\psi(x) + \overline{\psi}(x)\,\Big[\eta(x) + \alpha_{(0,0,1)}(x)\Big]\Big\} \\
&\qquad\qquad + \int_{x,y}\Big\{A_\mu(x)\Big[R^{\mu\nu}(x,y) + \alpha^{\mu\nu}_{(2,0,0)}(x,y)\Big]A_\nu(y) \\
&\qquad\qquad\quad + \overline{\psi}(x)\Big[K(x,y) + \alpha_{(0,1,1)}(x,y)\Big]\psi(y)\Big\} \\
&\qquad\qquad + \ldots\Bigg\}\Bigg)
\end{aligned} \tag{2.43}$$

where the dots indicate terms containing higher-order coefficients of the Taylor expansion of $f$. The important point now is to note that for *Gaussian* density operators $\rho_{\mathrm{Gauss}}$, i. e.

[27] In general, one should add bilinear sources for every combination of fields, i. e. in the case of QED for the pairs $(A_\mu, A_\nu)$, $(A_\mu,\psi)$, $(A_\mu,\overline{\psi})$, $(\psi,A_\mu)$, $(\psi,\psi)$, $(\psi,\overline{\psi})$, $(\overline{\psi},A_\mu)$, $(\overline{\psi},\psi)$, $(\overline{\psi},\overline{\psi})$ (where for each pair the first field is to be evaluated at $x$ and the second one at $y$). However, since we are only interested in the photon and fermion propagators, we can omit most of these combinations (see, however, Ref. [RS07]).

density operators whose Taylor expansion terminates at the second order[28], the dots do not indicate any missing terms, so that we conclude that

$$\begin{aligned} Z_{\rho_{\text{Gauss}}}[J,\eta,\overline{\eta},R,K] &= Z\Big[J+\alpha_{(1,0,0)},\eta+\alpha_{(0,1,0)},\overline{\eta}+\alpha_{(0,0,1)},R+\alpha_{(2,0,0)},K+\alpha_{(0,1,1)}\Big] \\ &= Z\Big[\widetilde{J},\widetilde{\eta},\widetilde{\overline{\eta}},\widetilde{R},\widetilde{K}\Big] \end{aligned} \tag{2.44}$$

where $Z = Z_{\rho_{\text{vac}}}$ is the vacuum generating functional.

The generating functional of connected correlation functions is defined by

$$Z_\rho[J,\eta,\overline{\eta},R,K] = \exp(\mathrm{i}\, W_\rho[J,\eta,\overline{\eta},R,K])\,, \tag{2.45}$$

and correspondingly we have

$$Z\Big[\widetilde{J},\widetilde{\eta},\widetilde{\overline{\eta}},\widetilde{R},\widetilde{K}\Big] = \exp\Big(\mathrm{i}\, W\Big[\widetilde{J},\widetilde{\eta},\widetilde{\overline{\eta}},\widetilde{R},\widetilde{K}\Big]\Big) \tag{2.46}$$

for Gaussian density operators. From this the 2PI effective action is obtained by a double Legendre transform with respect to the one-point as well as the two-point sources:

$$\begin{aligned} \Gamma_{2\text{PI}}[A,\overline{\psi},\psi,D,G] = W\Big[\widetilde{J},\widetilde{\eta},\widetilde{\overline{\eta}},\widetilde{R},\widetilde{K}\Big] &- \int_x \Big[A_\mu(x)\widetilde{J}^\mu(x) + \widetilde{\overline{\eta}}(x)\,\psi(x) + \overline{\psi}(x)\,\widetilde{\eta}(x)\Big] \\ &- \frac{1}{2}\int_{x,y}\Big\{\Big[A_\mu(x)A_\nu(y) + D_{\mu\nu}(x,y)\Big]\widetilde{R}^{\mu\nu}(x,y) \\ &\qquad + \Big[\overline{\psi}(x)\psi(y) + S(y,x)\Big]\widetilde{K}(x,y)\Big\} \end{aligned} \tag{2.47}$$

with $\widetilde{J} = \widetilde{J}(A,\overline{\psi},\psi,D,G)$ and similarly for the other external sources. The EoMs following from the 2PI effective action are then given by:

$$\left.\frac{\delta\Gamma_{2\text{PI}}[A,\overline{\psi},\psi,D,S]}{\delta A_\mu(x)}\right|_{\text{phys}} = \left.\Big[-\widetilde{J}^\mu(x) - \int_y \widetilde{R}^{\mu\nu}(x,y)A_\nu(y)\Big]\right|_{\text{phys}} = -\int_y \alpha_1^{(1,0,0)\mu\nu}(x,y)\,A_\nu(y)\,, \tag{2.48a}$$

$$\left.\frac{\delta\Gamma_{2\text{PI}}[A,\overline{\psi},\psi,D,S]}{\delta\overline{\psi}(x)}\right|_{\text{phys}} = \left.\Big[-\widetilde{\eta}(x) - \int_y \widetilde{K}(x,y)\psi(y)\Big]\right|_{\text{phys}} = -\int_y \alpha_1^{(0,1,0)}(x,y)\,\psi(y)\,, \tag{2.48b}$$

$$\left.\frac{\delta\Gamma_{2\text{PI}}[A,\overline{\psi},\psi,D,S]}{\delta\psi(x)}\right|_{\text{phys}} = \left.\Big[-\widetilde{\overline{\eta}}(x) - \int_y \overline{\psi}(x)\widetilde{K}(x,y)\Big]\right|_{\text{phys}} = -\int_y \overline{\psi}(x)\,\alpha_1^{(0,0,1)}(x,y)\,, \tag{2.48c}$$

$$\left.\frac{\delta\Gamma_{2\text{PI}}[A,\overline{\psi},\psi,D,S]}{\delta D_{\mu\nu}(x,y)}\right|_{\text{phys}} = -\frac{1}{2}\left.\widetilde{R}^{\mu\nu}(x,y)\right|_{\text{phys}} = -\frac{1}{2}\alpha_2^{(2,0,0)\mu\nu}(x,y)\,, \tag{2.48d}$$

$$\left.\frac{\delta\Gamma_{2\text{PI}}[A,\overline{\psi},\psi,D,S]}{\delta S(x,y)}\right|_{\text{phys}} = -\frac{1}{2}\left.\widetilde{K}(x,y)\right|_{\text{phys}} = -\frac{1}{2}\alpha_2^{(0,1,1)}(x,y)\,, \tag{2.48e}$$

[28] The most general Gaussian density matrix for a one-component scalar field can be found in Ref. [CHKM97].

where $\ldots|_{\text{phys}}$ means that all quantities are evaluated for vanishing external sources, i. e. $J = 0, \eta = 0, \overline{\eta} = 0, R = 0, K = 0$.

It now becomes obvious that it is exactly the *Gaussian* initial conditions which can be very efficiently described by the 2PI effective action, since the effect of Gaussian density operators can be completely absorbed into the external sources. The 2PI effective action for a Gaussian density operator therefore formally looks almost exactly the same as the 2PI effective action in vacuum, except for the fact that it is defined on a CTP and that, as we shall shortly see, the information contained in the density operator does show up again in the initial conditions for the EOMs (2.48).

# Chapter 3

# QED from the 2PI Effective Action

We will now present a convenient parametrization of the 2PI effective action for QED for vanishing field expectation values and derive the EOMs from it. They will turn out to be rather complicated, quantitatively as well as structurally. The quantitative complexity comes from the many components of the photon and fermion two-point functions.[1] For the photons, not all of them correspond to physical DOFs, though, and since the redundancy of DOFs employed in the description of gauge fields is one of the main features of gauge theories (and which finds an unusual manifestation in real-time formulations of them), we will have to say a few things about the physical DOFs of gauge theories first.

In order to render the EOMs tractable for a numerical study, we will then implement an initial state which exhibits certain symmetries while still retaining the features one is usually interested in. These symmetries will enable us to drastically reduce the number of independent components for the photons as well as for the fermions, and correspondingly the number of EOMs which have to be solved.

In order to obtain a system of equations which can indeed be solved, we have to provide initial conditions, which we will then present.

## 3.1 Equations of Motion

It can be shown that for vanishing field expectation values, the 2PI effective action can be parametrized as [CJT74]

$$\begin{aligned}
&\Gamma_{2\mathrm{PI}}[S,D] \\
&\quad = \frac{\mathrm{i}}{2}\operatorname{Tr}\ln\Big(D^{-1}\Big) + \frac{\mathrm{i}}{2}\operatorname{Tr}\Big(D_0^{-1}(D-D_0)\Big) - \mathrm{i}\operatorname{Tr}\ln\Big(S^{-1}\Big) - \mathrm{i}\operatorname{Tr}\Big(S_0^{-1}(S-S_0)\Big) + \Gamma_2[S,D]
\end{aligned} \tag{3.1}$$

where in a diagrammatic representation, $\Gamma_2[S,D]$ contains only 2PI diagrams, i. e. diagrams which remain connected when cutting two propagator lines, and the traces are over

[1] Altogether, we have 64 components: 16 for each of the spectral and statistical functions for photons and fermions, and the fermion components are even complex.

spacetime indices (discrete and continuous ones).[2]

The free inverse photon and fermion propagators read, respectively:

$$\mathrm{i}\left(D_0^{-1}\right)^{\mu\nu}(x,y) = \frac{\delta^2 S[A,\overline{\psi},\psi]}{\delta A_\mu(x)\delta A_\nu(y)} = \left[g^{\mu\nu}\Box_x - \left(1-\frac{1}{\xi}\right)\partial_x^\mu\partial_x^\nu\right]\delta^4(x-y)\,, \tag{3.2}$$

$$\mathrm{i}\left(S_0^{-1}\right)(x,y) = \frac{\delta^2 S[A,\overline{\psi},\psi]}{\delta\psi(x)\delta\overline{\psi}(y)} = \left(\mathrm{i}\,\gamma^\mu\partial_{x\mu} - m^{(\mathrm{f})}\right)\delta^4(x-y)\,. \tag{3.3}$$

Note that without the gauge fixing term, the photon propagator would be (proportional to) the four-dimensionally transverse projection operator and therefore noninvertible.[3]

Further note that the (Feynman or time-ordered) propagators $S$ and $D$ are *parameters* of the 2PI effective action. The *physical* propagators are those which extremize the 2PI effective action. This is in complete analogy to the 1PI effective action, where the field expectation value is a parameter, while its physical value extremizes the 1PI effective action.

In terms of the corresponding (Heisenberg) quantum field operators, the propagators are given by

$$D_{\mu\nu}(x,y) = \langle \mathrm{T} A_\mu(x) A_\nu(y)\rangle\,, \tag{3.4}$$

$$S(x,y) = \langle \mathrm{T}\psi(x)\overline{\psi}(y)\rangle\,, \tag{3.5}$$

for the photon and the fermion, respectively.

The EOMs can now easily be derived from the 2PI effective action. One obtains:

$$\frac{\delta\Gamma_{2\mathrm{PI}}[S,D]}{\delta D_{\mu\nu}(x,y)} = -\frac{\mathrm{i}}{2}\left(D^{-1}\right)^{\mu\nu}(x,y) + \frac{\mathrm{i}}{2}\left(D_0^{-1}\right)^{\mu\nu}(x,y) - \frac{\mathrm{i}}{2}\,\Pi^{\mu\nu}(x,y) = -\frac{1}{2}\,\alpha_2^{(2,0,0)\mu\nu}(x,y)\,, \tag{3.6}$$

$$\frac{\delta\Gamma_{2\mathrm{PI}}[S,D]}{\delta S(x,y)} = \mathrm{i}\,S^{-1}(x,y) - \mathrm{i}\,S_0^{-1}(x,y) + \mathrm{i}\,\Sigma(x,y) = -\frac{1}{2}\,\alpha_2^{(0,1,1)}(x,y)\,, \tag{3.7}$$

where we have defined the photon and fermion self-energy, respectively, as

$$\Pi^{\mu\nu}(x,y) = 2\,\mathrm{i}\,\frac{\delta\Gamma_2[S,D]}{\delta D_{\mu\nu}(x,y)}\,, \tag{3.8}$$

$$\Sigma(x,y) = -\,\mathrm{i}\,\frac{\delta\Gamma_2[S,D]}{\delta S(x,y)}\,, \tag{3.9}$$

i. e. (up to constant prefactors) as the variations of the 2PI part of the 2PI effective action with respect to the corresponding propagators.

[2] Truncations of the 2PI effective action are also often called $\Phi$-*derivable approximations* [Bay62], where the 2PI part $\Gamma_2$ is then denoted as $\Phi$.

[3] It is easy to see that it would annihilate longitudinal quantities like $\partial_\mu\Lambda$ (i. e. the difference between two gauge equivalent photon fields). Therefore, the not gauge-fixed free inverse photon propagator has vanishing eigenvalues.

The EOMs are of the same form as the DSEs for the propagators. Note, however, that these equations are completely self-consistent since the self-energies depend only on the propagators and on the *free* vertex. Therefore, no ansatz is needed for the full vertex function in order to close the equations as in the standard DSEs.

These equations are obviously not very well suited for studying the (time) evolution of the full propagators. It would be desirable for the EOMs to be differential equations with respect to time, reflecting the fact that they are evolution equations for the propagators. This can in fact be achieved by convolving the EOMs with the respecting full propagators, yielding

$$\left[g^{\mu\lambda}\Box_x - \left(1-\frac{1}{\xi}\right)\partial_x^\mu\partial_x^\lambda\right]D_{\lambda\nu}(x,y) + \int_z\left[\mathrm{i}\,\Pi^{\mu\lambda}(x,z) + \alpha_2^{(2,0,0)\mu\lambda}(x,z)\right]D_{\lambda\nu}(z,y) = \mathrm{i}\,\delta^\mu_\nu\delta^4(x-y)\,, \tag{3.10}$$

$$(\mathrm{i}\,\gamma^\mu\partial_{x\mu} - m^{(\mathrm{f})})S(x,y) + \int_z\left[\mathrm{i}\,\Sigma(x,z) + \alpha_2^{(0,1,1)}(x,z)\right]S(z,y) = \mathrm{i}\,\delta^4(x-y)\,. \tag{3.11}$$

From a practical point of view, these equations are still somewhat inconvenient, since the integrals are defined on a nontrivial contour (the CTP), meaning that in a calculation, one has to cope with quantities which are defined on two different branches (the branch from initial time to some given finite later time, and the one from there back to initial time). It would be much more convenient to have integrals extending over real intervals only. This can be achieved by decomposing the Feynman propagators into two new two-point functions, the so-called *statistical* and *spectral* functions. They correspond (up to a constant factor) to the expectation value of the anticommutator and of the commutator of two (Heisenberg) quantum field operators, respectively.[4] For the photon, these are

$$F^{(\mathrm{g})}_{\mu\nu}(x,y) = \frac{1}{2}\,\langle\{A_\mu(x), A_\nu(y)\}\rangle\,, \tag{3.12a}$$

$$\rho^{(\mathrm{g})}_{\mu\nu}(x,y) = \mathrm{i}\langle[A_\mu(x), A_\nu(y)]\rangle\,, \tag{3.12b}$$

while for the fermion, we have

$$F^{(\mathrm{f})}(x,y) = \frac{1}{2}\,\langle[\psi(x), \overline{\psi}(y)]\rangle\,, \tag{3.13a}$$

$$\rho^{(\mathrm{f})}(x,y) = \mathrm{i}\langle\{\psi(x), \overline{\psi}(y)\}\rangle\,. \tag{3.13b}$$

Making the time-ordering involved in the definition of the Feynman propagator explicit by expressing it through sign functions it is easy to see that they are connected with the corresponding Feynman propagators via

$$D_{\mu\nu}(x,y) = F^{(\mathrm{g})}_{\mu\nu}(x,y) - \frac{\mathrm{i}}{2}\,\mathrm{sgn}(x^0-y^0)\rho^{(\mathrm{g})}_{\mu\nu}(x,y)\,, \tag{3.14}$$

$$S(x,y) = F^{(\mathrm{f})}(x,y) - \frac{\mathrm{i}}{2}\,\mathrm{sgn}(x^0-y^0)\rho^{(\mathrm{f})}(x,y)\,. \tag{3.15}$$

[4]They are also sometimes called *Hadamard* and *(Pauli-)Jordan* propagators, respectively. Note that their definition regarding constant prefactors is not consistent in the literature. The spectral function, for instance, is also sometimes defined without the factor of i, like e. g. in Ref. [BI02].

In thermal equilibrium (with inverse temperature $\beta$), the statistical and spectral functions are not independent but related by the KMS periodicity condition $D_{\mu\nu}(x,y)|_{x^0=0} = D_{\mu\nu}(x,y)|_{x^0=-\mathrm{i}\,\beta}$ (and similarly for the fermion propagator), but out-of-equilibrium, they are completely independent in general, and hence their time evolution has to be studied separately.[5]

The EOMs for the statistical and spectral functions then follow from the EOMs for the Feynman propagators. Decomposing the self-energies into statistical and spectral parts in a similarly way as the propagators,[6]

$$\Pi^{\mu\nu}(x,y) = \Pi^{\mu\nu}_{(F)}(x,y) - \frac{\mathrm{i}}{2}\,\mathrm{sgn}(x^0-y^0)\Pi^{\mu\nu}_{(\rho)}(x,y)\,, \tag{3.16}$$

$$\Sigma(x,y) = \Sigma_{(F)}(x,y) - \frac{\mathrm{i}}{2}\,\mathrm{sgn}(x^0-y^0)\Sigma_{(\rho)}(x,y)\,, \tag{3.17}$$

we obtain for the photons:

$$\left[g^{\mu\lambda}\Box_x - \left(1-\frac{1}{\xi}\right)\partial_x^\mu\partial_x^\lambda\right]\rho^{(\mathrm{g})}_{\lambda\nu}(x,y) = \int_{y^0}^{x^0}\mathrm{d}z\,\Pi^{\mu\lambda}_{(\rho)}(x,z)\rho^{(\mathrm{g})}_{\lambda\nu}(z,y)\,, \tag{3.18a}$$

$$\begin{aligned}\left[g^{\mu\lambda}\Box_x - \left(1-\frac{1}{\xi}\right)\partial_x^\mu\partial_x^\lambda\right]F^{(\mathrm{g})}_{\lambda\nu}(x,y) &= \int_{t_0}^{x^0}\mathrm{d}z\,\Pi^{\mu\lambda}_{(\rho)}(x,z)F^{(\mathrm{g})}_{\lambda\nu}(z,y)\\ &\quad - \int_{t_0}^{y^0}\mathrm{d}z\left[\Pi^{\mu\lambda}_{(F)}(x,z) - \mathrm{i}\,\alpha_2^{(2,0,0)\mu\lambda}(x,z)\right]\rho^{(\mathrm{g})}_{\lambda\nu}(z,y)\,,\end{aligned} \tag{3.18b}$$

and for the fermions:[7]

$$(\mathrm{i}\,\gamma^\mu\partial_{x\mu} - m^{(\mathrm{f})})\rho^{(\mathrm{f})}(x,y) = \int_{y^0}^{x^0}\mathrm{d}z\,\Sigma_{(\rho)}(x,z)\rho^{(\mathrm{f})}(z,y)\,, \tag{3.20a}$$

$$\begin{aligned}(\mathrm{i}\,\gamma^\mu\partial_{x\mu} - m^{(\mathrm{f})})F^{(\mathrm{f})}(x,y) &= \int_{t_0}^{x^0}\mathrm{d}z\,\Sigma_{(\rho)}(x,z)F^{(\mathrm{f})}(z,y)\\ &\quad - \int_{t_0}^{y^0}\mathrm{d}z\left[\Sigma_{(F)}(x,z) - \mathrm{i}\,\alpha_2^{(0,1,1)}(x,z)\right]\rho^{(\mathrm{f})}(z,y)\,,\end{aligned} \tag{3.20b}$$

where $\int_{x^0}^{y^0}\mathrm{d}z := \int_{x^0}^{y^0}\mathrm{d}z^0\int\mathrm{d}^3z$. Note that the EOMs for the statistical functions depend explicitly on some initial time $t_0$, which without loss of generality can be set to zero.

[5] Note that the KMS condition is a boundary condition, and hence thermal equilibrium is a boundary value problem. In this respect, thermal equilibrium is qualitatively different from nonequilibrium, which is an initial value problem.

[6] There is no purely local contribution to the self-energies (which would lead to a spacetime-dependent mass shift) due to the absence of a tadpole in QED.

[7] If we write the memory integral on the right-hand side as $I^{(\mathrm{f})}_{(\rho)}(x,y)$, then $\rho^{(\mathrm{f})}$ satisfies the Klein-Gordon type equation

$$\left(\Box_x + m^{(\mathrm{f})2}\right)\rho^{(\mathrm{f})}(x,y) = -(\mathrm{i}\,\gamma^\mu\partial_{x\mu} + m^{(\mathrm{f})})I^{(\mathrm{f})}_{(\rho)}(x,y)\,, \tag{3.19}$$

and similarly for the statistical function.

In a real-time formulation as above, the integrals over time on the right-hand sides of the EoMs which encode the interactions are also called *memory integrals*, since they integrate over the entire history of the system, from the initial time to the current time.

Note again that the $\alpha_2$-terms encoding the initial density matrix are only nonvanishing at $t_0$. We have:

$$\begin{aligned}\left[g^{\mu\lambda}\Box_x - \left(1-\frac{1}{\xi}\right)\partial_x^\mu\partial_x^\lambda\right] F^{(\mathrm{g})}_{\lambda\nu}(x,y)\Big|_{x^0=y^0=t_0} \\ &= \mathrm{i}\lim_{y^0\to t_0}\int_{t_0}^{y^0}\mathrm{d}z^0\int\mathrm{d}^3z\,\alpha_2^{(2,0,0)\mu\lambda}(x,z)\Big|_{x^0=t_0}\rho^{(\mathrm{g})}_{\lambda\nu}(z,y)\\ &= \mathrm{i}\lim_{y^0\to t_0}(y^0-t_0)\int\mathrm{d}^3z\,\alpha_2^{(2,0,0)\mu\lambda}(x,z)\Big|_{x^0=t_0,z^0=y^0}\rho^{(\mathrm{g})}_{\lambda\nu}(z,y)\end{aligned}$$

and

$$\begin{aligned}(\mathrm{i}\,\gamma^\mu\partial_{x\mu}-m^{(\mathrm{f})})F^{(\mathrm{f})}(x,y)\Big|_{x^0=y^0=t_0} \\ &= \mathrm{i}\lim_{y^0\to t_0}\int_{t_0}^{y^0}\mathrm{d}z^0\int\mathrm{d}^3z\,\alpha_2^{(0,1,1)}(x,z)\Big|_{x^0=t_0}\rho^{(\mathrm{f})}(z,y)\\ &= \mathrm{i}\lim_{y^0\to t_0}(y^0-t_0)\int\mathrm{d}^3z\,\alpha_2^{(0,1,1)}(x,z)\Big|_{x^0=t_0,z^0=y^0}\rho^{(\mathrm{f})}(z,y)\,.\end{aligned}$$

It turns out that the terms on the right-hand side of the equations exactly encode the initial conditions. Compare this to the differential equation together with the initial conditions

$$y''(x)+y(x)=f(x)\,,\quad y(0)=y_0\,,\quad y'(0)=y_1\,,$$

which is equivalent to the single equation [Mat]

$$y''(x)+y(x)=\Theta(x)f(x)+y_0\,\delta'(x)+y_1\,\delta(x)\,,$$

where $\Theta$ is the Heaviside theta function[8]. In the same way, in the EoMs for the two-point functions, we can just as well omit the $\alpha_2$ altogether and instead provide the initial conditions for the respective two-point functions separately.

Note that the $\alpha_2$-terms appear only in the EoMs for the statistical function. According to what has just been stated about their connection to the initial values, this means that one is only free to choose initial conditions for the statistical functions, not for the spectral functions. This is because the initial conditions for the spectral functions are fixed by the equal-time commutation relations. We will elaborate on this in Sec. 3.4.

[8]The Heaviside theta function is defined as

$$\Theta(x)=\frac{1}{2}\left[1+\mathrm{sgn}(x)\right]=\begin{cases}0\,; & x<0\,,\\ 1/2\,; & x=0\,,\\ 1\,; & x>0\,.\end{cases}$$

Therefore, the EOMs are equivalent to the set of equations

$$\left[g^{\mu\lambda}\Box_x - \left(1-\frac{1}{\xi}\right)\partial_x^\mu\partial_x^\lambda\right]\rho^{(\mathrm{g})}_{\lambda\nu}(x,y) = \int_{y^0}^{x^0}\mathrm{d}z\,\Pi^{\mu\lambda}_{(\rho)}(x,z)\rho^{(\mathrm{g})}_{\lambda\nu}(z,y)\,, \tag{3.21a}$$

$$\left[g^{\mu\lambda}\Box_x - \left(1-\frac{1}{\xi}\right)\partial_x^\mu\partial_x^\lambda\right]F^{(\mathrm{g})}_{\lambda\nu}(x,y) = \int_{t_0}^{x^0}\mathrm{d}z\,\Pi^{\mu\lambda}_{(\rho)}(x,z)F^{(\mathrm{g})}_{\lambda\nu}(z,y) - \int_{t_0}^{y^0}\mathrm{d}z\,\Pi^{\mu\lambda}_{(F)}(x,z)\rho^{(\mathrm{g})}_{\lambda\nu}(z,y)\,, \tag{3.21b}$$

$$\rho^{(\mathrm{g})}_{\mu\nu}(x,y)\Big|_{x^0=y^0=t_0} = f^{(\mathrm{g})}_{1\,\mu\nu}(\boldsymbol{x},\boldsymbol{y})\,, \tag{3.21c}$$

$$\frac{\partial}{\partial x^0}\rho^{(\mathrm{g})}_{\mu\nu}(x,y)\Big|_{x^0=y^0=t_0} = f^{(\mathrm{g})}_{2\,\mu\nu}(\boldsymbol{x},\boldsymbol{y})\,, \tag{3.21d}$$

$$\frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}\rho^{(\mathrm{g})}_{\mu\nu}(x,y)\Big|_{x^0=y^0=t_0} = f^{(\mathrm{g})}_{3\,\mu\nu}(\boldsymbol{x},\boldsymbol{y})\,, \tag{3.21e}$$

$$F^{(\mathrm{g})}_{\mu\nu}(x,y)\Big|_{x^0=y^0=t_0} = g^{(\mathrm{g})}_{1\,\mu\nu}(\boldsymbol{x},\boldsymbol{y})\,, \tag{3.21f}$$

$$\frac{\partial}{\partial x^0}F^{(\mathrm{g})}_{\mu\nu}(x,y)\Big|_{x^0=y^0=t_0} = g^{(\mathrm{g})}_{2\,\mu\nu}(\boldsymbol{x},\boldsymbol{y})\,, \tag{3.21g}$$

$$\frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}F^{(\mathrm{g})}_{\mu\nu}(x,y)\Big|_{x^0=y^0=t_0} = g^{(\mathrm{g})}_{3\,\mu\nu}(\boldsymbol{x},\boldsymbol{y}) \tag{3.21h}$$

with functions $f^{(\mathrm{g})}_{1\,\mu\nu}, f^{(\mathrm{g})}_{2\,\mu\nu}, f^{(\mathrm{g})}_{3\,\mu\nu}$ which are fixed by the equal-time commutation relations and arbitrary functions $g^{(\mathrm{g})}_{1\,\mu\nu}, g^{(\mathrm{g})}_{2\,\mu\nu}, g^{(\mathrm{g})}_{3\,\mu\nu}$ for the photons, and

$$(\mathrm{i}\,\gamma^\mu\partial_{x\mu} - m^{(\mathrm{f})})\rho^{(\mathrm{f})}(x,y) = \int_{y^0}^{x^0}\mathrm{d}z\,\Sigma_{(\rho)}(x,z)\rho^{(\mathrm{f})}(z,y)\,, \tag{3.22a}$$

$$(\mathrm{i}\,\gamma^\mu\partial_{x\mu} - m^{(\mathrm{f})})F^{(\mathrm{f})}(x,y) = \int_{t_0}^{x^0}\mathrm{d}z\,\Sigma_{(\rho)}(x,z)F^{(\mathrm{f})}(z,y) - \int_{t_0}^{y^0}\mathrm{d}z\,\Sigma_{(F)}(x,z)\rho^{(\mathrm{f})}(z,y)\,, \tag{3.22b}$$

$$\rho^{(\mathrm{f})}(x,y)\Big|_{x^0=y^0=t_0} = f^{(\mathrm{f})}(\boldsymbol{x},\boldsymbol{y})\,, \tag{3.22c}$$

$$F^{(\mathrm{f})}(x,y)\Big|_{x^0=y^0=t_0} = g^{(\mathrm{f})}(\boldsymbol{x},\boldsymbol{y})\,, \tag{3.22d}$$

with a function $f^{(\mathrm{f})}$ which is fixed by the equal-time commutation relations and an arbitrary function $g^{(\mathrm{f})}$ for the fermions.

The sets of equations (3.21) and (3.22) constitute the initial value problem which determines the time evolution of the two-point correlation functions of the quantum fields of QED which at some initial time $t_0$ are in a state described by a Gaussian density operator.

## 3.2 Degrees of Freedom

The photon field is described by four real-valued functions, while photon two-point functions are correspondingly described by 16 real-valued functions and so on for higher correlation functions. Similarly, the fermion field is described by four complex-valued functions, while fermion two-point functions are described by 16 complex-valued functions and so on for higher correlation functions. However, not necessarily all of the components describe independent DOFs. There are two sources for the reduction of the independent number of DOFs: One is a possible symmetry of the initial state (vacuum or a state in- or out-of-equilibrium). This applies to photons as well as fermions. The second source is the gauge symmetry of the photons: Physical quantities which are related by a gauge transformation describe the same physics.

We will first discuss the physical, i. e. gauge invariant DOFs of photon and fermion, and afterwards the reduction of the independent number of DOFs for photons as well as for fermions by the symmetry of the initial state.

### 3.2.1 Physical Degrees of Freedom of the Fermion

According to (2.8), the fermion fields are not gauge invariant and therefore cannot be physical. The same is true for correlation functions containing fermion fields, like the fermion propagator (except for equal spacetime points). It is, however, possible to define a gauge invariant fermion field according to [Dir58]

$$\psi_{\text{phys}}(x) = \psi(x)\exp\left(-\mathrm{i}\,e\,\frac{\partial^i A_i(x)}{\boldsymbol{\nabla}^2}\right)\,, \tag{3.23}$$

and correspondingly for the Dirac conjugate. It can easily be checked that $\psi_{\text{phys}}(x)$ is gauge invariant, since under a gauge transformation, we have

$$\begin{aligned}\psi_{\text{phys}}(x) \mapsto \quad & \mathrm{e}^{\mathrm{i}\,\Lambda(x)}\,\psi(x)\exp\left(-\mathrm{i}\,e\,\frac{\partial^i\left[A_i(x)-\frac{1}{e}\,\partial_i\Lambda(x)\right]}{\boldsymbol{\nabla}^2}\right)\\ &= \mathrm{e}^{\mathrm{i}\,\Lambda(x)}\,\psi(x)\exp\left(-\mathrm{i}\,e\,\frac{\partial^i A_i(x)}{\boldsymbol{\nabla}^2}-\mathrm{i}\,\Lambda(x)\right)\\ &= \mathrm{e}^{\mathrm{i}\,\Lambda(x)}\,\psi_{\text{phys}}(x)\,\mathrm{e}^{-\mathrm{i}\,\Lambda(x)}\\ &= \psi_{\text{phys}}(x)\,.\end{aligned}$$

Although it is hence possible to define a gauge invariant fermion, $\psi_{\text{phys}}$ is an awkward object to work with since it is highly nonlocal due to the appearance of the differential operator in the exponential. In most cases, it is therefore easier to work with the original, gauge noninvariant object $\psi(x)$.

### 3.2.2 Physical Degrees of Freedom of the Photon

We have to discriminate two types of physical DOFs: fundamental ones and effective ones. Fundamental DOFs are those which can be attributed to a single photon, without reference to any kind of interaction with its environment. They are the only DOFs given in vacuum. Effective DOFs, however, are due to the interaction with a background medium and hence usually do not exist in vacuum. Effective DOFs therefore depend on a possible background medium, and in particular on its symmetries. While it is usually clear what the fundamental DOFs of a theory are, it is often not so easy to identify the effective DOFs since they could be generated by complicated interactions.

In gauge theories, however, the situation is complicated by the fact that not all DOFs are physical, so we further have a discrimination into physical and unphysical DOFs. In a sense, this is the very essence of gauge theories. While the photon field $A_\mu$ is described by four real numbers[9], it is well known that a photon has only two DOFs, corresponding to its two spin states or polarization directions transverse to its direction of propagation.[10] Therefore, two of the DOFs contained in $A_\mu$ must be unphysical. Let us Fourier transform the photon field to momentum space and split it according to

$$A_\mu(p) = A_{\perp\mu}(p) + A_{\parallel\mu}(p) \tag{3.24}$$

with

$$A_{\perp\mu}(p) = \mathsf{P}_{\perp\mu}^{\ \ \nu}(p)A_\nu(p) = \left(g_\mu^\nu - \frac{p_\mu p^\nu}{p^2}\right)A_\nu(p)\,, \quad A_{\parallel\mu}(p) = \mathsf{P}_{\parallel\mu}^{\ \ \nu}(p) = \frac{p_\mu p^\nu}{p^2}A_\nu(p)\,. \tag{3.25}$$

Here,

$$\mathsf{P}_{\perp\mu\nu}(p) = g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \quad \text{and} \quad \mathsf{P}_{\parallel\mu\nu}(p) = \frac{p_\mu p_\nu}{p^2} \tag{3.26}$$

are the four-dimensionally transverse and longitudinal projection operators, respectively.[11] Since $\mathsf{P}_\perp \mathsf{P}_\parallel = 0$, Eq. (3.24) is an orthogonal decomposition of the photon field.

Let us now consider the gauge transformation of the photon field,

$$A_\mu(p) \mapsto A_\mu^\Lambda(p) = A_\mu(p) - \frac{\mathrm{i}}{e}p_\mu\Lambda(p)\,.$$

The transverse and longitudinal parts of the photon field transform according to

$$A_{\perp\mu}(p) \mapsto A_{\perp\mu}(p)\,, \quad A_{\parallel\mu}(p) \mapsto A_{\parallel\mu}(p) - \frac{\mathrm{i}}{e}p_\mu\Lambda(p)\,,$$

[9] For the sake of simplicity, we only talk about expectation values, i. e. numbers, here so that we do not need to consider operators. The fact that due to Lorentz invariance, there cannot be a nonvanishing expectation value of the photon field, does not matter for our reasoning here.

[10] For a massive vector particle, one could also have polarization *along* its direction of propagation; for a massless particle like the photon, however, which propagates with the speed of light, this is not possible.

[11] A projection operator $\mathsf{P}$ is an operator which is idempotent, i. e. $\mathsf{P}^2 = \mathsf{P}$. It follows that it is not invertible unless it is the unit operator.

i. e. a gauge transformation effectively acts on the longitudinal component of the photon field only, while the transverse component is gauge invariant.[12] It is then clear that the physical DoFs of the photon must be contained in the transverse part $A_{\perp\mu}$ only, since in particular they have to be gauge invariant. From this point of view, gauge symmetry is a manifestation of the fact that we chose a description of the photon which contains superfluous, unphysical DoFs in the first place. It is, however, not the whole four-dimensionally transverse part of the photon which corresponds to the physical DoFs since the four-dimensionally transverse projection operator has rank 3, so there is still one unphysical DoF left, and it has to be removed by fixing the gauge.

Before we turn to that, let us briefly consider the free photon propagator. In a linear covariant gauge, the free photon propagator reads

$$D_{0\mu\nu}(p) = -\frac{\mathrm{i}}{p^2}\left[g_{\mu\nu} - (1-\xi)\,\frac{p_\mu p_\nu}{p^2}\right], \tag{3.27}$$

so that one has

$$\begin{aligned}\mathrm{i}\,D_{0\perp\mu\nu}(p) &= \mathsf{P}_{\perp}{}^{\rho}_{\mu}(p)\mathsf{P}_{\perp}{}^{\sigma}_{\nu}(p)\,\mathrm{i}\,D_{0\,\rho\sigma}(p) = \frac{1}{p^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right),\\ \mathrm{i}\,D_{0\parallel\mu\nu}(p) &= \mathsf{P}_{\parallel}{}^{\rho}_{\mu}(p)\mathsf{P}_{\parallel}{}^{\sigma}_{\nu}(p)\,\mathrm{i}\,D_{0\,\rho\sigma}(p) = \frac{\xi}{p^2}\,\frac{p_\mu p_\nu}{p^2},\end{aligned}$$

or, getting rid of the tensor structure,

$$\begin{aligned}\mathrm{i}\,D_{0\perp}(p) &= \frac{1}{3}\,\mathsf{P}_{\perp}^{\mu\nu}(p)\,\mathrm{i}\,D_{0\mu\nu}(p) = \frac{1}{p^2},\\ \mathrm{i}\,D_{0\parallel}(p) &= \mathsf{P}_{\parallel}^{\mu\nu}(p)\,\mathrm{i}\,D_{0\mu\nu}(p) = \frac{\xi}{p^2},\end{aligned}$$

i. e. the four-dimensionally longitudinal part explicitly depends on the gauge fixing parameter. This is another confirmation for the fact that the physical DoFs must be contained in the transverse part.[13]

It is clear that one cannot hope to find a Lorentz invariant description of the physical DoFs since "transverse to its direction of propagation" is not a Lorentz invariant statement.[14] The direction of propagation is just the direction of the spatial momentum vector $\boldsymbol{p}$. Since for now, we are only talking about the fundamental DoFs of the photon,

[12] To be more precise: A gauge transformation acts on $A_{\perp\mu}$ as the identity transformation, i. e. in a trivial way.

[13] In fact, in terms of the four-dimensionally transverse and four-dimensionally longitudinal components of the photon, the photon part of the effective classical action reads:

$$S_{\mathrm{g}}[A] + S_{\mathrm{gf}}^{\xi}[A] = \int_x \left[\frac{1}{2}\,(\partial_\mu A_{\perp\nu})(\partial^\mu A_\perp^\nu) - \frac{1}{2\xi}\,(\partial^\mu A_{\parallel\mu})^2\right].$$

[14] Of course, "transverse" and "direction" are to be understood as spatial terms here.

we can safely assume that we have transformed all quantities to momentum space.[15] We can then easily write down a projection operator which projects onto a plane transverse to $\boldsymbol{p}$:[16]

$$\mathsf{P}_{\perp}^{\mathrm{T}\,ij}(\boldsymbol{p}) = g^{ij} + \frac{p^i\,p^j}{\boldsymbol{p}^2}\,, \tag{3.28}$$

since $\mathsf{P}_{\perp\,i}^{\mathrm{T}\,j}(\boldsymbol{p})\,p_j = 0$. Although not a covariant object, the projector can be written in a "pseudo-covariant" way by introducing a collection of four numbers $n^\mu$ constituting an axis which explicitly breaks Lorentz symmetry [KG06, Wel82]:[17]

$$\mathsf{P}_{\perp}^{\mathrm{T}\,\mu\nu}(p,n) = g^{\mu\nu} - \frac{p^\mu p^\nu + p^2 n^\mu n^\nu - (n\cdot p)(p^\mu n^\nu + p^\nu n^\mu)}{p^2 - (n\cdot p)^2}\,, \tag{3.29}$$

or componentwise:

$$\begin{aligned} \mathsf{P}_{\perp}^{\mathrm{T}\,00}(p_0,\boldsymbol{p}) = \mathsf{P}_{\perp}^{\mathrm{T}\,i0}(p_0,\boldsymbol{p}) = \mathsf{P}_{\perp}^{\mathrm{T}\,0i}(p_0,\boldsymbol{p}) = 0\,,\\ \mathsf{P}_{\perp}^{\mathrm{T}\,ij}(p_0,\boldsymbol{p}) = g^{ij} + \frac{p_i p_j}{\boldsymbol{p}^2}\,. \end{aligned} \tag{3.30}$$

Obviously, the only nonvanishing components are the purely spatial ones, and they are independent of $p_0$. This is an important point, since out-of-equilibrium, we cannot even Fourier transform with respect to temporal components.

For the physical photon field $A_{\mathrm{phys}\,\mu}$, we then have:[18]

$$A_{\mathrm{phys}\,\mu}(p,n) = \mathsf{P}_{\perp\,\mu}^{\mathrm{T}\,\nu}(p,n)A_\nu(p)\,, \tag{3.31}$$

so that

$$A_{\mathrm{phys}\,0}(p,n) = 0\,, \qquad A_{\mathrm{phys}\,i}(p,n) = \left(g_i^j + \frac{p_i\,p^j}{\boldsymbol{p}^2}\right)A_j(p)\,. \tag{3.32}$$

For the sake of definiteness, consider a photon propagating in $x^3$-direction, so that $(p_\mu) = (1,0,0,1)|\boldsymbol{p}|$. Then:

$$A_{\mathrm{phys}\,1}(p,n) = A_1(p)\,, \quad A_{\mathrm{phys}\,2}(p,n) = A_2(p)\,, \quad A_{\mathrm{phys}\,3}(p,n) = 0\,.$$

[15] This will not be the case for the effective DOFs, since they depend on a background medium. Depending on its symmetries, it might not be possible to do a full Fourier transform to momentum space.

[16] Actually, the projection operator depends not on $\boldsymbol{p}$, but only on its direction $\hat{\boldsymbol{p}} := \boldsymbol{p}/|\boldsymbol{p}|$, so it would be more precise to write

$$\mathsf{P}_{\perp}^{\mathrm{T}\,ij}(\hat{\boldsymbol{p}}) = g^{ij} + \hat{p}^i\,\hat{p}^j\,.$$

[17] Note that a dependence on $p$ and $n$ is equivalent to a dependence on $\boldsymbol{p}$ and $p_0$ (i. e. they are treated as independent quantities). Usually, we write $(p,n)$ when we employ a (pseudo-)covariant notation and $(p_0,\boldsymbol{p})$ when we consider the temporal and spatial components separately.

Further note that formally

$$\lim_{n\to 0} \mathsf{P}_{\perp}^{\mathrm{T}}(p,n) = \mathsf{P}_{\perp}(p)\,,$$

i. e. in the vacuum limit (to be more precise, $n$ does not vanish, but does not even exist in vacuum), the spatially transverse projector becomes the four-dimensionally longitudinal projector.

[18] Note that $A_{\mathrm{phys}\mu}$ is *not* a Lorentz (co-)vector due to its dependence on $n^\mu$.

Similarly, the physical photon propagator is then given by

$$D_{\mathrm{phys}\,\mu\nu}(p,n) = \mathsf{P}^{\mathrm{T}\,\rho}_{\perp\mu}(p,n)\mathsf{P}^{\mathrm{T}\,\sigma}_{\perp\nu}(p,n)D_{\rho\sigma}(p)\,, \tag{3.33}$$

so that

$$\begin{aligned} D_{\mathrm{phys}\,00}(p,n) &= D_{\mathrm{phys}\,i0}(p,n) = D_{\mathrm{phys}\,0i}(p,n) = 0\,,\\ D_{\mathrm{phys}\,ij}(p,n) &= \left(g_i^k + \frac{p_i\,p^k}{\boldsymbol{p}^2}\right)\left(g_j^l + \frac{p_j\,p^l}{\boldsymbol{p}^2}\right)D_{kl}(p)\,. \end{aligned} \tag{3.34}$$

For the free case, one finds:

$$\mathrm{i}\,D_{0\,\mathrm{phys}\,ij}(p,n) = \frac{1}{p^2}\left(g_{ij} + \frac{p_i\,p_j}{\boldsymbol{p}^2}\right) = \frac{1}{p^2}\,\mathsf{P}^{\mathrm{T}}_{\perp ij}(p,n)\,, \tag{3.35}$$

or, getting rid of the tensor structure[19],

$$\mathrm{i}\,D_{0\,\mathrm{phys}\,\mathrm{T}}(p) = \frac{1}{2}\,\mathsf{P}^{\mathrm{T}\,ij}_{\perp}(p,n)\,\mathrm{i}\,D_{0\,\mathrm{phys}\,ij}(p,n) = \frac{1}{p^2}\,. \tag{3.36}$$

Up to the sign, this is the free propagator of a massless scalar particle. In particular, it does not depend on the preferred direction $n^\mu$.

If there is enough energy in the system so that (real) particles can be created, however, an additional DoF emerges. In contrast to the three-dimensionally transverse DoF, which is fundamental, the additional DoF is a collective one. Since the speed of a photon propagating in an environment of particles is effectively reduced due to its interaction with the particles, it behaves itself like a massive particle.[20] According to what has been stated above, it is then possible for it to be polarized along its direction of propagation. This amounts to a longitudinal polarization, and the corresponding DoF is known as the *plasmon* in the literature [KG06, WH96]. It turns out that there is in fact a second projection operator which is four-dimensionally transverse (and hence yields a gauge invariant quantity when applied to a photon field) and three-dimensionally longitudinal, namely [KG06, Wel82]:[21]

$$\mathsf{P}^{\mathrm{L}\,\mu\nu}_{\perp}(p,n) = g^{\mu\nu} - \frac{p^\mu\,p^\nu}{p^2} - \mathsf{P}^{\mathrm{T}\,\mu\nu}_{\perp}(p,n)\,, \tag{3.38}$$

[19] The factor $1/2$ is due to the fact that there are two (degenerate) transverse DoFs.

[20] It is not a fundamental mass which would break gauge invariance, but an effective one which subsumes the effects of the interaction with the environment. In fact, it is not a fundamental particle, but a "quasiparticle".

[21] Note that formally

$$\lim_{n\to 0}\mathsf{P}^{\mathrm{L}}_{\perp}(p,n) = 0\,, \tag{3.37}$$

i. e. in the vacuum limit, the spatially longitudinal projector and hence the spatially longitudinal DoF vanish. This affirms the interpretation of the spatially longitudinal DoF as a dynamical effect which is due to the interaction with the system.

or componentwise:

$$\begin{aligned}
\mathsf{P}^{\mathrm{L}}_{\perp 00}(p_0,\boldsymbol{p}) &= -\frac{\boldsymbol{p}^2}{p^2}\,,\\
\mathsf{P}^{\mathrm{L}}_{\perp i0}(p_0,\boldsymbol{p}) = \mathsf{P}^{\mathrm{L}}_{\perp 0i}(p_0,\boldsymbol{p}) &= -\frac{p_i p_0}{p^2} = -\frac{p_0|\boldsymbol{p}|}{p^2}\frac{p_i}{|\boldsymbol{p}|}\,,\\
\mathsf{P}^{\mathrm{L}}_{\perp ij}(p_0,\boldsymbol{p}) &= -\frac{p_0^2}{p^2}\frac{p_i p_j}{\boldsymbol{p}^2}\,.
\end{aligned} \tag{3.39}$$

Unfortunately, however, the longitudinal projection operator does not share the nice property of the transverse one of having only spatial nonvanishing components and being independent of $p_0$. There is even a $p_0$-dependence in the denominators. In position space, the longitudinal projection operator would therefore be an object which is nonlocal in time, which is not practicable to work with.

In the free case, we obviously have

$$\mathrm{i}\,D_{0\,\mathrm{phys}\,\mathrm{L}}(p,n) = \mathsf{P}^{\mathrm{L}\,ij}_{\perp}(p,n)\,\mathrm{i}\,D_{0\,\mathrm{phys}\,ij}(p,n) = 0 \tag{3.40}$$

since $\mathsf{P}^{\mathrm{T}}_{\perp}$ and $\mathsf{P}^{\mathrm{L}}_{\perp}$ are orthogonal to each other, i. e. $\mathsf{P}^{\mathrm{T}}_{\perp}\mathsf{P}^{\mathrm{L}}_{\perp} = 0$. This is in accordance with the fact that in vacuum (or in a system without interactions) there does not exist a longitudinal physical DoF. However, in linear covariant gauges, we have

$$\mathrm{i}\,D_{0\perp}^{\mathrm{L}}(p) = \mathsf{P}^{\mathrm{L}\,ij}_{\perp}(p,n)\,\mathrm{i}\,D_{0\,ij}(p) = \frac{1}{p^2} = \mathrm{i}\,D_{0\perp}^{\mathrm{T}}(p)\,, \tag{3.41}$$

i. e. even in vacuum, there is a propagating longitudinal DoF. This is a clear indication that linear covariant gauges cannot be physical in the sense that only physical DoFs are described, and the longitudinal DoF in vacuum has to be removed by hand.

In the end, one is interested in the gauge invariant, potentially physical quantities

$$D^{\mathrm{T}}_{\perp}(p) = \mathsf{P}^{\mathrm{T}\,\mu\nu}_{\perp}D_{\mu\nu}(p) \quad \text{and} \quad D^{\mathrm{L}}_{\perp}(p) = \mathsf{P}^{\mathrm{L}\,\mu\nu}_{\perp}D_{\mu\nu}(p)\,, \tag{3.42}$$

and it seems to be obvious to discard the unphysical DoFs altogether. In vacuum and thermal equilibrium, this can in fact easily be done by projecting onto the gauge invariant DoFs. If time-translation invariance is not given, however, one would have to work with projection operators in real time. This is no problem for the transverse one, since it only depends on spatial quantities. The longitudinal projection operator, however, does depend on temporal quantities (which even appear in the denominator), which in real time would translate to a timelike nonlocality. Out-of-equilibrium, it is therefore impractical to obtain the physical DoFs by projection.

One therefore has to follow a different approach: One solves the EoMs for *all* DoFs, including the unphysical ones. Thereby it is guaranteed that no information is lost, and the question has been shifted to the extraction of the physical DoFs *after* the solution to the EoMs has been obtained.

In fact, it is usually not even necessary to solve the EOMs for virtually *all* DOFs since the number of independent DOFs is reduced by the symmetries of the initial state under consideration. The question for the minimum number of DOFs that have to be evolved is what we will turn to next.

First, however, let us conclude that we have seen that it is possible to define QED in terms of physical fields $(A_{\mathrm{phys}\mu}, \overline{\psi}_{\mathrm{phys}}, \psi_{\mathrm{phys}})$ only (see also Ref. [Ste84]). The price to pay is that relativistic covariance is lost since $A_{\mathrm{phys}\mu}$ is not a Lorentz vector, and that locality is lost since $\overline{\psi}_{\mathrm{phys}}$ and $\psi_{\mathrm{phys}}$ are nonlocal. It is therefore usually more convenient to work with the original fields $(A_\mu, \overline{\psi}, \psi)$, and this is also what we will do.

### 3.2.3 Spatially Homogeneous, Isotropic System

In this section, we will specialize the so far general Gaussian initial state to a spatially homogeneous, isotropic one, in order to reduce the number of independent components.

#### Symmetries

It is instructive to start with the most symmetric state, the (Minkowski) vacuum, and then successively reduce the symmetries until we arrive at a system which has sufficiently few symmetries so that it captures all the features we are interested in, but is still as symmetric as possible in order to simplify its treatment. The vacuum is homogeneous and isotropic with respect to spacetime. Spacetime homogeneity corresponds to invariance under spacetime translations, while spacetime isotropy corresponds to invariance under Lorentz transformations. The symmetry group of the vacuum is hence given by the Poincaré group $\mathrm{SO}(3,1) \times \mathbb{R}^4$, meaning that it is invariant under simultaneous Lorentz transformations and spacetime translations. A generic two-point function $f$ can therefore depend only on the difference of its two spacetime arguments, $x - y$, i. e. on four real numbers. One can then Fourier transform and trade the relative spacetime position for momentum, $x - y \to p$.

Thermal equilibrium corresponds to a spatially homogeneous, isotropic state together with time translation invariance. Spatial homogeneity corresponds to invariance under spatial translations, while spatial isotropy corresponds to invariance under rotations. The symmetry group of a thermal equilibrium state is hence given by the group $\mathrm{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}$, meaning that it is invariant under simultaneous rotations and spatial translations and *separately* under time translations. Since the medium breaks the symmetry between space and time explicitly, spatial and temporal dependencies become independent, so $(x^\mu) \to (x^0, \boldsymbol{x})$. A generic two-point function $f$ can therefore depend on the difference of its time and space arguments *separately*, $(x^0 - y^0, \boldsymbol{x} - \boldsymbol{y})$, i. e. on four real numbers (like in vacuum). One can then do a Fourier transform and trade time differences for energy, and spatial position differences for spatial momentum, $(x^0 - y^0, \boldsymbol{x} - \boldsymbol{y}) \to (E, \boldsymbol{p})$.

Since we are interested in a time evolution, time translation invariance has to be broken, and we arrive at a system with symmetry group $\mathrm{SO}(3) \times \mathbb{R}^3$. A generic two-point function $f$ can therefore depend on the difference of its space arguments and on both time arguments (which are independent now[22]), $(x^0, y^0, \boldsymbol{x}-\boldsymbol{y})$, i. e. on five real numbers. One can then do a partial Fourier transform with respect to space and trade the spatial position difference for spatial momentum, $(x^0, y^0, \boldsymbol{x}-\boldsymbol{y}) \to (x^0, y^0; \boldsymbol{p})$.

The statements made above are summarized Table 3.1.

| initial state | symmetry group | functional dependence before | functional dependence after | # |
|---|---|---|---|---|
| vacuum | $\mathrm{SO}(3,1) \times \mathbb{R}^4$ | $\lvert x-y\rvert$ | $\lvert p\rvert$ | 1 |
| thermal equilibrium | $\mathrm{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}$ | $(\lvert x^0-y^0\rvert, \lvert\boldsymbol{x}-\boldsymbol{y}\rvert)$ | $(E, \lvert\boldsymbol{p}\rvert)$ | 2 |
| spatially homogeneous and isotropic | $\mathrm{SO}(3) \times \mathbb{R}^3$ | $(x^0, y^0; \lvert\boldsymbol{x}-\boldsymbol{y}\rvert)$ | $(x^0, y^0; \lvert\boldsymbol{p}\rvert)$ | 3 |
| spatially homogeneous | $\mathbb{R}^3$ | $(x^0, y^0; \boldsymbol{x}-\boldsymbol{y})$ | $(x^0, y^0; \boldsymbol{p})$ | 5 |
| general | – (no symmetry) | $(x, y)$ | $(x, y)$ | 8 |

Table 3.1: Comparison of systems with different symmetries, where "before" refers to the arguments in position space, and "after" refers to the arguments after a possible (partial) Fourier transformation in order to reduce the number of arguments.

The given symmetry also affects possible internal components. For instance, the number of independent internal components is less in vacuum than in thermal equilibrium, as is to be expected. This is because symmetry transformations in vacuum can only act on spacetime components in a symmetric way, while in thermal equilibrium, they can act differently on the temporal and spatial components.

The question of the independent components can also be phrased in a different way: Which quantities are naturally, i. e. *a priori*, given that can be used to construct a basis for the given quantities?[23] Let us illustrate this with a few examples:

**Vacuum** In vacuum, there are two naturally given tensorial quantities: The metric $g_{\mu\nu}$, and the derivative operator $\partial_\mu$, which becomes (up to a constant factor) the momentum four-vector $p_\mu$ after Fourier transformation. A basis for any tensor in vacuum (in Fourier space) can therefore only consist of the metric, the momentum, and tensor products of them. Constant quantities vanish when acting with the derivative operator on them (or, equivalently, are independent of momentum), so they may only involve the metric.

- Constant vector: There is no naturally given constant vector in vacuum.
- Constant rank-two tensor: A basis is given by

$$B = \{g_{\mu\nu}\},$$

[22] Alternatively, one could say that there is not only a dependence on $x^0 - y^0$, but also on $x^0 + y^0$.

[23] Of course, one can always construct a basis out of unit vectors along arbitrary (independent) directions, for instance. However, this would introduce a kind of arbitrariness since they are not physical.

so every constant rank-two tensor $M_{\mu\nu}$ in vacuum must be proportional to the metric:

$$M_{\mu\nu} = g_{\mu\nu} M_{(1)} .$$

In particular, every constant rank-two tensor field in vacuum is symmetric. Altogether, the independent numbers have been reduced from the 16 numbers $(M_{\mu\nu})$ to the single number $M_{(1)}$. An example of a corresponding physical quantity is the vacuum energy-momentum tensor $T_{\mu\nu} = \Lambda g_{\mu\nu}$, where $\Lambda$ is the cosmological constant.

- Vector field: A basis is given by

$$\widetilde{B} = \{p^\mu\} ,$$

or, normalizing to unity and making the basis dimensionless,

$$B = \left\{ \frac{p^\mu}{|p|} \right\} ,$$

so that the vector field must be parallel to momentum[24],

$$v^\mu(p) = \frac{p^\mu}{|p|} v_{(1)}(|p|) ,$$

Therefore, the number of independent quantities has been reduced from four functions $v^\mu$ depending on four numbers $p = (p_\mu)$ to a single function $v_{(1)}$ depending on a single number $|p|$. Note that $v_{(1)}$ and $|p|$ are Lorentz scalars.

- Rank-two tensor field: An obvious basis is given by the metric and the tensor product of two momentum vectors,

$$\widetilde{B} = \{g_{\mu\nu}, p_\mu \, p_\nu\} .$$

A more convenient choice, however, is the basis

$$B = \left\{ g_{\mu\nu} - \frac{p_\mu \, p_\nu}{p^2}, \frac{p_\mu \, p_\nu}{p^2} \right\} ,$$

since, in addition to being normalized, the basis tensors are also orthogonal to each other (one of them projects along the direction of momentum, and the other projects onto the plane transverse to it). Then:

$$M_{\mu\nu}(p) = \left( g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right) M_{(1)}(|p|) + \frac{p_\mu p_\nu}{p^2} M_{(2)}(|p|) .$$

Therefore, the number of independent quantities has been reduced from 16 functions $(M_{\mu\nu})$ depending on four numbers $p = (p_\mu)$ to two functions $M_{(1)}, M_{(2)}$ depending on a single number $|p|$. Note that $M_{(1)}$ and $M_{(2)}$ are Lorentz scalars.

[24] Equivalently, one could write the argument as $p^2$ instead of $|p|$, as is found in many textbooks.

**Thermal equilibrium:** In a system which is in thermal equilibrium, there are three naturally given physical quantities: In addition the metric and the derivative operator or four-momentum as in vacuum, there is also the four-velocity $n = (n^\mu)$ of the system which breaks the Lorentz symmetry of the vacuum explicitly. This not only affects the tensorial structure of physical quantities, but (unless they are constant) also their arguments: In a spatially homogeneous, isotropic system, quantities depend not on the four-momentum $p = (p_\mu)$, but on $p$ and $n \cdot p$ (or, equivalently, on $(n \cdot p)\, n$ and $p^\mu - (n \cdot p)\, n$, or on $p_0$ and $\boldsymbol{p}$) separately. If we choose $n^\mu = \delta^\mu_0$, we have

$$(n \cdot p)\, n = \begin{pmatrix} p_0 \\ 0 \end{pmatrix} , \quad p - (n \cdot p)\, n = \begin{pmatrix} 0 \\ \boldsymbol{p} \end{pmatrix} .$$

and

$$\sqrt{(n \cdot p)^2 - p^2} = |\boldsymbol{p}| .$$

Note that $n \cdot p$ and $\sqrt{(n \cdot p)^2 - p^2}$ are scalars with respect to SO(3)-rotations.

- Constant vector: An obvious basis is given by

  $$B = \{n^\mu\} = \{\delta^\mu_0\} ,$$

  so that

  $$v^\mu = n^\mu v_{(1)} = \delta^\mu_0 v_{(1)} ,$$

  or

  $$v^0 = v_{(1)} , \quad v^i = 0 .$$

  Therefore, the number of independent quantities has been reduced from four to one. A physical example is an electromagnetic current $J^\mu = \rho\, n^\mu$: In an isotropic system, there can only be a nonvanishing charge density $\rho$, but no nonvanishing spatial currents $J^i = \rho\, n^i$.
- Constant rank-two tensor: An obvious basis is given by the metric and the tensor product of two medium velocity vectors,

  $$\widetilde{B} = \{g_{\mu\nu}, n_\mu n_\nu\} = \{\delta^0_\mu \delta^0_\nu + \delta^i_\mu \delta^j_\nu\, g_{ij}, \delta^0_\mu \delta^0_\nu\} .$$

  A more convenient one, however, is given by

  $$B = \{g_{\mu\nu} - n_\mu n_\nu, n_\mu n_\nu\} = \{\delta^i_\mu \delta^j_\nu\, g_{ij}, \delta^0_\mu \delta^0_\nu\} ,$$

  which is orthogonal (one basis tensor projects along the direction of the four-velocity of the system and the other one transverse to it), so that

  $$M_{\mu\nu} = (g_{\mu\nu} - n_\mu n_\nu) M_{(1)} + n_\mu n_\nu M_{(2)} = \delta^i_\mu \delta^j_\nu\, g_{ij} M_{(1)} + \delta^0_\mu \delta^0_\nu M_{(2)} ,$$

  or

  $$M_{00} = M_{(2)} , \quad M_{ij} = g_{ij} M_{(1)} , \quad M_{i0} = M_{0i} = 0 .$$

  Therefore, the number of independent numbers has been reduced from the 16 numbers $(M_{\mu\nu})$ to the two numbers $M_{(1)}, M_{(2)}$. Note that in vacuum, $M_{(1)} = M_{(2)}$.

- Vector field: An obvious basis is given by

$$\widetilde{B} = \{n^\mu, p^\mu\} = \{\delta^\mu_0, \delta^\mu_0 p^0 + \delta^\mu_i p^i\}\,.$$

A more convenient choice, however, is the basis

$$B = \{n^\mu, \hat{p}^\mu - (n\cdot\hat{p})\, n^\mu\} = \{\delta^\mu_0, \delta^\mu_i \hat{p}^i\}\,,$$

where we have defined $\hat{p} = p/\sqrt{(n\cdot p)^2 - p^2} = p/|\boldsymbol{p}|$.[25] Then:

$$\begin{aligned} v^\mu(p) &= n^\mu\, v_{(1)}\Big(n\cdot p, \sqrt{(n\cdot p)^2 - p^2}\Big) \\ &\qquad + \Big[\hat{p}^\mu - (n\cdot\hat{p})\, n^\mu\Big] v_{(2)}\Big(n\cdot p, \sqrt{(n\cdot p)^2 - p^2}\Big) \end{aligned}$$

or

$$v^\mu(p_0, \boldsymbol{p}) = \delta^\mu_0 v_{(1)}(p_0, |\boldsymbol{p}|) + \delta^\mu_i \,\frac{p_i}{|\boldsymbol{p}|}\, v_{(2)}(p_0, |\boldsymbol{p}|)\,,$$

so that

$$v^0(p) = v_{(1)}(p_0, |\boldsymbol{p}|)\,, \quad v^i(p) = \frac{p^i}{|\boldsymbol{p}|}\, v_{(2)}(p_0, |\boldsymbol{p}|)\,.$$

The number of independent quantities has hence been reduced from the four functions $(v^\mu)$ to the two functions $v_{(1)}, v_{(2)}$ and from the four numbers $p = (p_\mu)$ to the two numbers $n\cdot p = p_0$ and $\sqrt{(n\cdot p)^2 - p^2} = |\boldsymbol{p}|$.
Therefore, of the four potentially independent functions, only two remain. Note that in vacuum, $v_{(1)} = v_{(2)}$.

- Rank-two tensor field: An obvious basis is given by the metric, the tensor product of a momentum vector with itself and a medium velocity vector with itself together with the symmetrized and antisymmetrized tensor product of a momentum vector with a medium velocity vector,[26]

$$\begin{aligned} \widetilde{B} &= \{g_{\mu\nu}, p_\mu\, p_\nu, n_\mu\, n_\nu, p_\mu\, n_\nu + p_\nu\, n_\mu, p_\mu\, n_\nu - p_\nu\, n_\mu\} \\ &= \{\delta^0_\mu \delta^0_\nu + \delta^i_\mu \delta^j_\nu\, g_{ij}, (\delta^0_\mu + \delta^i_\mu p_i)(\delta^0_\nu + \delta^j_\nu p_j), \delta^0_\mu \delta^0_\nu, \\ &\quad (\delta^0_\mu p_0 + \delta^i_\mu p_i)\delta^0_\nu + (\delta^0_\nu p_0 + \delta^i_\nu p_i)\delta^0_\mu, (\delta^0_\mu p_0 + \delta^i_\mu p_i)\delta^0_\nu - (\delta^0_\nu p_0 + \delta^i_\nu p_i)\delta^0_\mu\}\,. \end{aligned}$$

[25] The reason why we do not choose to orthogonalize the basis by defining $\hat{p} = p/|p|$ instead will become clear later.

[26] One advantage of this basis is that it easily decomposes into a basis for symmetric rank-two tensors and antisymmetric rank-two tensors, $\widetilde{B} = \widetilde{B}_{\mathrm{sym}} \cup \widetilde{B}_{\mathrm{asym}}$ (and $\widetilde{B}_{\mathrm{sym}} \cap \widetilde{B}_{\mathrm{asym}} = \emptyset$), with $\widetilde{B}_{\mathrm{sym}} = \{g_{\mu\nu}, p_\mu\, p_\nu, n_\mu\, n_\nu, p_\mu\, n_\nu + p_\nu\, n_\mu\}$, i. e. symmetric second-rank tensor fields depend on four independent functions, and $\widetilde{B}_{\mathrm{asym}} = \{p_\mu\, n_\nu - p_\nu\, n_\mu\}$, i. e. antisymmetric second-rank tensor fields depend on a single independent function.

A more convenient basis, however, is given by

$$
\begin{aligned}
B = \Big\{ & n^\mu\, n^\nu, n^\mu[\hat{p}^\nu - (n\cdot\hat{p})\, n^\nu], [\hat{p}^\mu - (n\cdot\hat{p})\, n^\mu]\, n^\nu, \\
& - [\hat{p}^\mu - (n\cdot\hat{p})\, n^\mu][\hat{p}^\nu - (n\cdot\hat{p})\, n^\nu], \\
& g^{\mu\nu} + \hat{p}^\mu\hat{p}^\nu - [1-(n\cdot\hat{p})^2]\, n^\mu n^\nu - (n\cdot\hat{p})(\hat{p}^\mu n^\nu + \hat{p}^\nu n^\mu)\Big\} \\
= & \left\{ \delta^\mu_0\delta^\nu_0, \delta^\mu_0\delta^\nu_i\, \frac{p^i}{|\boldsymbol{p}|}, \delta^\mu_i\delta^\nu_0\, \frac{p^i}{|\boldsymbol{p}|}, \delta^\mu_i\delta^\nu_j\, \frac{p^ip^j}{\boldsymbol{p}^2}, \delta^\mu_i\delta^\nu_j \left( g^{ij} + \frac{p^ip^j}{\boldsymbol{p}^2}\right)\right\},
\end{aligned}
$$

so that

$$
\begin{aligned}
& M^{\mu\nu}(p) \\
& = n^\mu n^\nu M_{(1)}\Big(n\cdot p, \sqrt{(n\cdot p)^2 - p^2}\Big) + n^\mu\Big[\hat{p}^\nu - (n\cdot\hat{p})\, n^\nu\Big]\, M_{(2)}\Big(n\cdot p, \sqrt{(n\cdot p)^2}\Big) \\
& \quad + \Big[\hat{p}^\mu - (n\cdot\hat{p})\, n^\mu\Big]\, n^\nu M_{(3)}\Big(n\cdot p, \sqrt{(n\cdot p)^2}\Big) \\
& \quad - \Big[\hat{p}^\mu - (n\cdot\hat{p})\, n^\mu\Big]\Big[\hat{p}^\nu - (n\cdot\hat{p})\, n^\nu\Big] M_{(4)}\Big(n\cdot p, \sqrt{(n\cdot p)^2 - p^2}\Big) \\
& \quad + \Big\{ g^{\mu\nu} + \hat{p}^\mu\hat{p}^\nu - \Big[1-(n\cdot\hat{p})^2\Big]\, n^\mu n^\nu - (n\cdot\hat{p})(\hat{p}^\mu n^\nu + \hat{p}^\nu n^\mu)\Big\} \\
& \qquad \cdot M_{(5)}\Big(n\cdot p, \sqrt{(n\cdot p)^2 - p^2}\Big)\,,
\end{aligned}
$$

or

$$
\begin{aligned}
M^{\mu\nu}(p_0,\boldsymbol{p}) = {} & \delta^\mu_0\delta^\nu_0 M_{(1)}(p_0,|\boldsymbol{p}|) + \delta^\mu_0\delta^\nu_i\, \frac{p^i}{|\boldsymbol{p}|}\, M_{(2)}(p_0,|\boldsymbol{p}|) + \delta^\mu_i\delta^\nu_0\, \frac{p^j}{|\boldsymbol{p}|}\, M_{(3)}(p_0,|\boldsymbol{p}|) \\
& - \delta^\mu_i\delta^\nu_j\, \frac{p^ip^j}{\boldsymbol{p}^2} M_{(4)}(p_0,|\boldsymbol{p}|) + \delta^\mu_i\delta^\nu_j \left( g^{ij} + \frac{p^ip^j}{\boldsymbol{p}^2}\right) M_{(5)}(p_0,|\boldsymbol{p}|)
\end{aligned}
$$

with

$$
\begin{aligned}
M^{00}(p_0,\boldsymbol{p}) &= M_{(1)}(p_0,|\boldsymbol{p}|)\,, \\
M^{i0}(p_0,\boldsymbol{p}) &= \frac{p^i}{|\boldsymbol{p}|}\, M_{(2)}(p_0,|\boldsymbol{p}|)\,, \quad M^{0i}(p_0,\boldsymbol{p}) = \frac{p^i}{|\boldsymbol{p}|}\, M_{(3)}(p_0,|\boldsymbol{p}|)\,, \\
M^{ij}(p_0,\boldsymbol{p}) &= -\frac{p^ip^j}{\boldsymbol{p}^2}\, M_{(4)}(p_0,|\boldsymbol{p}|) + \left( g^{ij} + \frac{p^ip^j}{\boldsymbol{p}^2}\right) M_{(5)}(p_0,|\boldsymbol{p}|)\,.
\end{aligned}
$$

Therefore, of the 16 potentially independent functions ($M^{\mu\nu}$), only the five functions $M_{(i)}$ ($i = 1,\ldots,5$) remain. Note that $M^{00}$ is an O(3)-scalar, $M^{i0}$ and $M^{0i}$ are O(3)-vectors, and $M^{ij}$ is a second-rank O(3)-tensor, while the functions $M_{(i)}$ are O(3)-scalars.
Further note that the purely spatial components $M^{ij}$ are written in a basis of projection operators.

For a symmetric second-rank tensor field, instead of two independent functions $M_{(2)}$ and $M_{(3)}$, there is only one independent function $M_{(2+3)}$ with

$$M^{i0}(p_0,\boldsymbol{p}) = M^{0i}(p) = \frac{p^i}{|\boldsymbol{p}|}\, M_{(2+3)}(p_0,|\boldsymbol{p}|)\,.$$

For an antisymmetric second-rank tensor field, there is only one independent function $M_{(2-3)}$ altogether with

$$M^{i0}(p_0,\boldsymbol{p}) = -M^{0i}(p) = \frac{p^i}{|\boldsymbol{p}|}\, M_{(2-3)}(p_0,|\boldsymbol{p}|)\,,$$

and all other functions vanish identically.

Formally, the vacuum expressions are obtained in the limit $n \to 0$ from the medium expressions, and the constant expressions are obtained in the limit $p \to 0$ from the field expressions. It is clear that the construction of basis tensors in states with less symmetry quickly becomes rather involved.

It becomes clear now why we have chosen bases different from the "naive" ones: In the "improved" bases, the basis tensors are independent of $p_0$, and the whole $p_0$-dependence is contained in the scalar functions. We can therefore use the same bases if we drop time-translation invariance (i. e. temporal homogeneity), and the only difference will be that in the scalar functions, we have to replace $p_0$ by $(x^0, y^0)$, i. e. the functions depend on $x^0$ and $y^0$ separately instead of only on their difference $x^0 - y^0$. The number of independent functions, however, remains the same, the only difference being that they depend not on just two numbers $(p_0, |\boldsymbol{p}|)$, but on three numbers $(x^0, y^0, |\boldsymbol{p}|)$.[27]

## Photons

It immediately follows from the above considerations that the photon spectral function can be decomposed according to:[28]

$$\begin{aligned}\rho^{(\mathrm{g})}_{\mu\nu}(x^0,y^0;\boldsymbol{p}) &= \delta^0_\mu\delta^0_\nu\rho^{(\mathrm{g})}_{\mathrm{S}}(x^0,y^0;|\boldsymbol{p}|) + \delta^0_\mu\delta^i_\nu\,\frac{p_i}{|\boldsymbol{p}|}\,\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(x^0,y^0;|\boldsymbol{p}|) + \delta^i_\mu\delta^0_\nu\,\frac{p_i}{|\boldsymbol{p}|}\,\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(x^0,y^0;|\boldsymbol{p}|)\\ &\quad + \delta^i_\mu\delta^j_\nu\left[\left(g_{ij}+\frac{p_ip_j}{\boldsymbol{p}^2}\right)\rho^{(\mathrm{g})}_{\mathrm{T}}(x^0,y^0;|\boldsymbol{p}|) - \frac{p_ip_j}{\boldsymbol{p}^2}\,\rho^{(\mathrm{g})}_{\mathrm{L}}(x^0,y^0;|\boldsymbol{p}|)\right]\,, \qquad (3.43)\end{aligned}$$

where we have defined the functions $\rho^{(\mathrm{g})}_{\mathrm{S}}$, $\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}$, $\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}$, $\rho^{(\mathrm{g})}_{\mathrm{T}}$, and $\rho^{(\mathrm{g})}_{\mathrm{L}}$, which we refer to as the "scalar", the "type 1 vector", the "type 2 vector"[29], the "transverse" and the "longitudi-

[27] Where each of these numbers is a scalar under rotations.

[28] In fact, the vector components are not actually independent since from $\rho^{(\mathrm{g})}_{\mu\nu}(x,y) = -\rho^{(\mathrm{g})}_{\nu\mu}(y,x)$, it follows that $\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(x^0,y^0;p) = -\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(y^0,x^0;p)$ (and similarly for the statistical function). However, in spite of this identity, we will treat them as independent. The reason is to be found in the implementation of the EoMs on a computer (see also App. E): In order to save memory, one conveniently only stores times $(x^0, y^0)$ for which $y^0 \leq x^0$. In this case, the above identity obviously cannot be used.

[29] In thermal equilibrium, the type 1 and type 2 vector components are identical, but out-of-equilibrium, they are in general not.

nal" component, respectively, or as "isotropic" components collectively.[30] An analogous decomposition of course holds for the statistical function as well.

Note that

$$\rho_{\perp}^{(\mathrm{g})\mathrm{T}}(x^0, y^0; |\boldsymbol{p}|) = \rho_{\mathrm{T}}^{(\mathrm{g})}(x^0, y^0; |\boldsymbol{p}|) \,. \tag{3.44}$$

In a spatially homogeneous, isotropic system, the number of independent components of the photon two-point functions is hence reduced from 16 to five compared to a system without any symmetries.

### Fermions

Fermionic two-point functions have 16 discrete components as well, which are, however, complex in general. A basis for the fermionic two-point functions can be constructed out of the gamma matrices.[31] The fermion spectral function (in fact each fermionic two-point function) can be written as

$$\rho^{(\mathrm{f})}(x,y) = \rho_{\mathrm{S}}^{(\mathrm{f})}(x,y) + \mathrm{i}\,\gamma_5 \rho_{\mathrm{P}}^{(\mathrm{f})}(x,y) + \gamma^\mu \rho_{\mathrm{V}\mu}^{(\mathrm{f})}(x,y) + \gamma^\mu \gamma_5 \rho_{\mathrm{A}\mu}^{(\mathrm{f})}(x,y) + \frac{1}{2}\,\sigma^{\mu\nu} \rho_{\mathrm{T}\mu\nu}^{(\mathrm{f})}(x,y) \tag{3.45}$$

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[30] A complete basis is hence given by $\{\mathsf{e}_{\mathrm{S}\,\mu\nu}^{(\mathrm{g})}, \mathsf{e}_{\mathrm{V}_1\,\mu\nu}^{(\mathrm{g})}, \mathsf{e}_{\mathrm{V}_2\,\mu\nu}^{(\mathrm{g})}, \mathsf{e}_{\mathrm{T}\,\mu\nu}^{(\mathrm{g})}, \mathsf{e}_{\mathrm{L}\,\mu\nu}^{(\mathrm{g})}\}$ with

$$\mathsf{e}_{\mathrm{S}\,\mu\nu}^{(\mathrm{g})} = \mathsf{P}_{\mathrm{S}}^{(\mathrm{g})}\,\delta_\mu^0\delta_\nu^0\,, \quad \mathsf{e}_{\mathrm{V}_1\,\mu\nu}^{(\mathrm{g})} = \mathsf{P}_{\mathrm{V}\,i}^{(\mathrm{g})}\,\delta_\mu^i\delta_\nu^0\,, \quad \mathsf{e}_{\mathrm{V}_2\,\mu\nu}^{(\mathrm{g})} = \mathsf{P}_{\mathrm{V}\,i}^{(\mathrm{g})}\,\delta_\mu^0\delta_\nu^i\,, \quad \mathsf{e}_{\mathrm{T}\,\mu\nu}^{(\mathrm{g})} = \mathsf{P}_{\mathrm{T}\,ij}^{(\mathrm{g})}\,\delta_\mu^i\delta_\nu^j\,, \quad \mathsf{e}_{\mathrm{L}\,\mu\nu}^{(\mathrm{g})} = \mathsf{P}_{\mathrm{L}\,ij}^{(\mathrm{g})}\,\delta_\mu^i\delta_\nu^j$$

with

$$\mathsf{P}_{\mathrm{S}}^{(\mathrm{g})} = 1\,, \quad \mathsf{P}_{\mathrm{V}\,i}^{(\mathrm{g})} = \mathrm{i}\,\frac{p_i}{p}\,, \quad \mathsf{P}_{\mathrm{T}\,ij}^{(\mathrm{g})} = g_{ij} + \frac{p_i p_j}{p^2}\,, \quad \mathsf{P}_{\mathrm{L}\,ij}^{(\mathrm{g})} = -\frac{p_i p_j}{p^2}\,.$$

These quantities have the following properties:

$$\mathsf{P}_{\mathrm{S}}^{(\mathrm{g})\,2} = 1 = \mathsf{P}_{\mathrm{S}}^{(\mathrm{g})}\,, \quad \mathsf{P}_{\mathrm{V}\,i}^{(\mathrm{g})}\,\mathsf{P}_{\mathrm{V}}^{(\mathrm{g})\,i} = 1\,, \quad \mathsf{P}_{\mathrm{V}}^{(\mathrm{g})\,i}\,\mathsf{P}_{\mathrm{V}}^{(\mathrm{g})\,j} = \mathsf{P}_{\mathrm{L}}^{(\mathrm{g})\,ij}\,, \quad \mathsf{P}_{\mathrm{T/L}\,i}^{(\mathrm{g})\quad k}\mathsf{P}_{\mathrm{T/L}\,kj}^{(\mathrm{g})} = \mathsf{P}_{\mathrm{T/L}\,ij}^{(\mathrm{g})}\,,$$
$$\mathsf{P}_{\mathrm{T}\,ij}^{(\mathrm{g})}\mathsf{P}_{\mathrm{T}}^{(\mathrm{g})\,ij} = 2\,, \quad \mathsf{P}_{\mathrm{L}\,ij}^{(\mathrm{g})}\mathsf{P}_{\mathrm{L}}^{(\mathrm{g})\,ij} = 1\,, \quad \mathsf{P}_{\mathrm{T}\,i}^{(\mathrm{g})k}\mathsf{P}_{\mathrm{L}\,kj}^{(\mathrm{g})} = 0\,,$$

i. e. $\mathsf{P}_{\mathrm{T}}^{(\mathrm{g})}$ and $\mathsf{P}_{\mathrm{L}}^{(\mathrm{g})}$ are projection operators.

[31] In fact, $\{\mathsf{e}_{\mathrm{S}}^{(\mathrm{f})}, \mathsf{e}_{\mathrm{P}}^{(\mathrm{f})}, \mathsf{e}_{\mathrm{V}}^{(\mathrm{f})\,\mu}, \mathsf{e}_{\mathrm{A}}^{(\mathrm{f})\,\mu}, \mathsf{e}_{\mathrm{T}}^{(\mathrm{f})\,\mu\nu}\}$ with

$$\mathsf{e}_{\mathrm{S}}^{(\mathrm{f})} = \mathbf{1}\,, \quad \mathsf{e}_{\mathrm{P}}^{(\mathrm{f})} = \gamma_5\,, \quad \mathsf{e}_{\mathrm{V}}^{(\mathrm{f})\,\mu} = \gamma^\mu\,, \quad \mathsf{e}_{\mathrm{A}}^{(\mathrm{f})\,\mu} = \gamma_5\gamma^\mu\,, \quad \mathsf{e}_{\mathrm{T}}^{(\mathrm{f})\,\mu\nu} = \sigma^{\mu\nu}$$

is an orthogonal basis with respect to the scalar product $\langle A, B\rangle = \mathrm{tr}(AB)/4$. Setting $\mathsf{e}_{\mathrm{T}}^{(\mathrm{f})\,\mu\nu} \to \mathsf{e}_{\mathrm{T}}^{(\mathrm{f})\,\mu\nu}/2$, it could even be turned into an orthonormal basis.

$\mathsf{e}_{\mathrm{S}}^{(\mathrm{f})}$ transforms as a scalar, $\mathsf{e}_{\mathrm{P}}^{(\mathrm{f})}$ as a pseudoscalar, $\mathsf{e}_{\mathrm{V}}^{(\mathrm{f})\,\mu}$ as a vector, $\mathsf{e}_{\mathrm{A}}^{(\mathrm{f})\,\mu}$ as a pseudovector (or axial vector), and $\mathsf{e}_{\mathrm{T}}^{(\mathrm{f})\,\mu\nu}$ as a second-rank tensor under Lorentz transformations. I. e., given a Lorentz transformation $\Lambda$,

$$\mathsf{e}_{\mathrm{S}}^{(\mathrm{f})} \mapsto \mathsf{e}_{\mathrm{S}}^{(\mathrm{f})}\,, \quad \mathsf{e}_{\mathrm{P}}^{(\mathrm{f})} \mapsto \det(\Lambda)\,\mathsf{e}_{\mathrm{P}}^{(\mathrm{f})}\,, \quad \mathsf{e}_{\mathrm{V}}^{(\mathrm{f})\,\mu} \mapsto \Lambda^\mu{}_\nu \mathsf{e}_{\mathrm{V}}^{(\mathrm{f})\,\nu}\,, \quad \mathsf{e}_{\mathrm{A}}^{(\mathrm{f})\,\mu} \mapsto \det(\Lambda)\,\Lambda^\mu{}_\nu \mathsf{e}_{\mathrm{A}}^{(\mathrm{f})\,\nu}\,, \quad \mathsf{e}_{\mathrm{T}}^{(\mathrm{f})\,\mu\nu} \mapsto \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma \mathsf{e}_{\mathrm{T}}^{(\mathrm{f})\,\rho\sigma}\,.$$

Note that there is no pseudotensor representation since $\det(\Lambda)^2 = 1$.

with $\gamma_5 = \mathrm{i}\,\gamma^0\gamma^1\gamma^2\gamma^3$, $\sigma^{\mu\nu} = \mathrm{i}[\gamma^\mu,\gamma^\nu]/2$, and[32]

$$\begin{aligned}
\rho_{\mathrm{S}}^{(\mathrm{f})}(x,y) &= \frac{1}{4}\,\mathrm{tr}(\rho^{(\mathrm{f})}(x,y))\,,\\
\rho_{\mathrm{P}}^{(\mathrm{f})}(x,y) &= \frac{1}{4\,\mathrm{i}}\,\mathrm{tr}(\gamma_5\rho^{(\mathrm{f})}(x,y))\,,\\
\rho_{\mathrm{V}}^{(\mathrm{f})\mu}(x,y) &= \frac{1}{4}\,\mathrm{tr}(\gamma^\mu\rho^{(\mathrm{f})}(x,y))\,,\\
\rho_{\mathrm{A}}^{(\mathrm{f})\mu}(x,y) &= \frac{1}{4}\,\mathrm{tr}(\gamma_5\gamma^\mu\rho^{(\mathrm{f})}(x,y))\,,\\
\rho_{\mathrm{T}}^{(\mathrm{f})\mu\nu}(x,y) &= \frac{1}{4}\,\mathrm{tr}(\sigma^{\mu\nu}\rho^{(\mathrm{f})}(x,y))\,.
\end{aligned} \tag{3.46}$$

Due to the fact that the gamma matrices are Lorentz vectors, the components, which we refer to as "Lorentz components", have a definite behavior under Lorentz transformations, which makes it easy to exploit spacetime symmetries [PS95]. Under Lorentz transformations, the Lorentz components transform as a scalar, a pseudoscalar, a vector, an axial vector, and a second-rank tensor, respectively. Due to the CP symmetry of QED, the pseudoscalar and axial vector components vanish identically.

We then have:

$$\begin{aligned}
\rho^{(\mathrm{f})}(x^0,y^0;\boldsymbol{p}) &= \rho_{\mathrm{S}}^{(\mathrm{f})}(x^0,y^0;\boldsymbol{p}) + \gamma^\mu\rho_{\mathrm{V}\mu}^{(\mathrm{f})}(x^0,y^0;\boldsymbol{p}) + \frac{1}{2}\sigma^{\mu\nu}\rho_{\mathrm{T}\mu\nu}^{(\mathrm{f})}(x^0,y^0;\boldsymbol{p})\\
&= \rho_{\mathrm{S}}^{(\mathrm{f})}(x^0,y^0;|\boldsymbol{p}|) + \gamma^\mu\left[\delta_\mu^0\rho_{\mathrm{V}}^{(\mathrm{f})0}(x^0,y^0;|\boldsymbol{p}|) + \delta_\mu^i\frac{p_i}{|\boldsymbol{p}|}\rho_{\mathrm{V}}^{(\mathrm{f})}(x^0,y^0;\boldsymbol{p})\right]\\
&\quad + \frac{1}{2}\sigma^{\mu\nu}(\delta_\mu^i\delta_\nu^0 - \delta_\mu^0\delta_\nu^i)\frac{p_i}{|\boldsymbol{p}|}\rho_{\mathrm{T}}^{(\mathrm{f})}(x^0,y^0;|\boldsymbol{p}|)\\
&= \rho_{\mathrm{S}}^{(\mathrm{f})}(x^0,y^0;|\boldsymbol{p}|) + \mathrm{i}\,\gamma^0\widetilde{\rho}_{\mathrm{V}}^{(\mathrm{f})0}(x^0,y^0;|\boldsymbol{p}|)\\
&\quad - \boldsymbol{\gamma}\cdot\frac{\boldsymbol{p}}{|\boldsymbol{p}|}\,\rho_{\mathrm{V}}^{(\mathrm{f})}(x^0,y^0;|\boldsymbol{p}|) + \mathrm{i}\,\gamma^0\boldsymbol{\gamma}\cdot\frac{\boldsymbol{p}}{|\boldsymbol{p}|}\,\rho_{\mathrm{T}}^{(\mathrm{f})}(x^0,y^0;|\boldsymbol{p}|)
\end{aligned} \tag{3.47}$$

with $\mathrm{i}\,\widetilde{\rho}_{\mathrm{V}}^{(\mathrm{f})0} = \rho_{\mathrm{V}}^{(\mathrm{f})0}$[33], and where we have used that $\sigma^{i0} = \mathrm{i}\,\gamma^i\gamma^0$.

Note that each of the components $\rho_{\mathrm{S}}^{(\mathrm{f})}$, $\widetilde{\rho}_{\mathrm{V}}^{(\mathrm{f})0}$, $\rho_{\mathrm{V}}^{(\mathrm{f})}$, and $\rho_{\mathrm{T}}^{(\mathrm{f})}$ is a scalar under spatial rotations and hence only depends on the modulus of the spatial momentum. Further, all four components are real instead of complex.

In a spatially homogeneous, isotropic system, the number of independent components of the fermion two-point functions is hence reduced from 16 complex to four real ones compared to a system without any symmetries.

Altogether, the total number of real numbers needed to specify the independent components of the photon statistical and spectral functions and the fermion statistical and spectral functions has hence been reduced from $2\cdot16+2\cdot32 = 96$ (in general fermions

[32] One can define a scalar product $\langle A,B\rangle = \mathrm{tr}(AB)$, where $A, B$ are Dirac matrices. Then one can define projection operators $\mathsf{P}_{\mathrm{S}} = \langle \mathbf{1},\cdot\rangle$, $\mathsf{P}_{\mathrm{V}}^\mu = \langle\gamma^\mu,\cdot\rangle$ etc., and write, for instance, $\rho_{\mathrm{S}} = \mathsf{P}_{\mathrm{S}}\rho$.

[33] The reason for this definition is that $\rho_{\mathrm{V}}^{(\mathrm{f})0}$ is purely imaginary, so that $\widetilde{\rho}_{\mathrm{V}}^{(\mathrm{f})0}$ is real; see also App. D.

have 16 complex and hence 32 real internal components) to $2 \cdot 5 + 2 \cdot 4 = 18$, i. e. by a factor of more than 5. Further, to specify the mode of a quantity at a given time, only three numbers $(x^0, y^0; p)$ are needed instead of eight numbers $(x, y)$. We stress, however, that these simplifying assumptions are in no way necessary and in particular are not an approximation to the dynamics; they only reduce the number of independent components, thereby allowing us to focus on our main concern, i. e. on the evolution in time.

### Equations of Motion for a Spatially Homogeneous, Isotropic System

In the following, we will only work with the scalar components introduced above for photons and fermions and hence denote the modulus of the spatial momentum simply by $p$.

**Photon Equations of Motion** In terms of the components introduced above, the EoMs for the photon spectral function then read with $(x^0, y^0) \to (t, t')$:

$$\begin{aligned}
&\left(\frac{1}{\xi}\frac{\partial^2}{\partial t^2} + p^2\right) \rho_{\mathrm{S}}^{(\mathrm{g})}(t,t';p) + \left(1 - \frac{1}{\xi}\right) p \, \frac{\partial}{\partial t} \widetilde{\rho}_{\mathrm{V}_1}^{(\mathrm{g})}(t,t';p) \\
&\qquad = \int_{t'}^{t} \mathrm{d}t'' \left[ \Pi_{(\rho)\mathrm{S}}(t,t'';p)\, \rho_{\mathrm{S}}^{(\mathrm{g})}(t'',t';p) + \widetilde{\Pi}_{(\rho)\mathrm{V}_2}(t,t'';p)\, \widetilde{\rho}_{\mathrm{V}_1}^{(\mathrm{g})}(t'',t';p) \right], \quad (3.48\mathrm{a}) \\
&\left(\frac{\partial^2}{\partial t^2} + \frac{p^2}{\xi}\right) \widetilde{\rho}_{\mathrm{V}_1}^{(\mathrm{g})}(t,t';p) + \left(1 - \frac{1}{\xi}\right) p \, \frac{\partial}{\partial t} \rho_{\mathrm{S}}^{(\mathrm{g})}(t,t';p) \\
&\qquad = \int_{t'}^{t} \mathrm{d}t'' \left[ \widetilde{\Pi}_{(\rho)\mathrm{V}_1}(t,t'';p)\, \rho_{\mathrm{S}}^{(\mathrm{g})}(t'',t';p) + \Pi_{(\rho)\mathrm{L}}(t,t'';p)\, \widetilde{\rho}_{\mathrm{V}_1}^{(\mathrm{g})}(t'',t';p) \right], \quad (3.48\mathrm{b}) \\
&\left(\frac{1}{\xi}\frac{\partial^2}{\partial t^2} + p^2\right) \widetilde{\rho}_{\mathrm{V}_2}^{(\mathrm{g})}(t,t';p) + \left(1 - \frac{1}{\xi}\right) p \, \frac{\partial}{\partial t} \rho_{\mathrm{L}}^{(\mathrm{g})}(t,t';p) \\
&\qquad = \int_{t'}^{t} \mathrm{d}t'' \left[ \Pi_{(\rho)\mathrm{S}}(t,t'';p)\, \widetilde{\rho}_{\mathrm{V}_2}^{(\mathrm{g})}(t'',t';p) + \widetilde{\Pi}_{(\rho)\mathrm{V}_2}(t,t'';p)\, \rho_{\mathrm{L}}^{(\mathrm{g})}(t'',t';p) \right], \quad (3.48\mathrm{c}) \\
&\left(\frac{\partial^2}{\partial t^2} + \frac{p^2}{\xi}\right) \rho_{\mathrm{L}}^{(\mathrm{g})}(t,t';p) + \left(1 - \frac{1}{\xi}\right) p \, \frac{\partial}{\partial t} \widetilde{\rho}_{\mathrm{V}_2}^{(\mathrm{g})}(t,t';p) \\
&\qquad = \int_{t'}^{t} \mathrm{d}t'' \left[ \widetilde{\Pi}_{(\rho)\mathrm{V}_1}(t,t'';p)\, \widetilde{\rho}_{\mathrm{V}_2}^{(\mathrm{g})}(t'',t';p) + \Pi_{(\rho)\mathrm{L}}(t,t'';p)\, \rho_{\mathrm{L}}^{(\mathrm{g})}(t'',t';p) \right], \quad (3.48\mathrm{d}) \\
&\left(\frac{\partial^2}{\partial t^2} + p^2\right) \rho_{\mathrm{T}}^{(\mathrm{g})}(t,t';p) = \int_{t'}^{t} \mathrm{d}t''\, \Pi_{(\rho)\mathrm{T}}(t,t'';p)\, \rho_{\mathrm{T}}^{(\mathrm{g})}(t'',t';p)\,, \quad (3.48\mathrm{e})
\end{aligned}$$

and the EoMs for the statistical function read:

$$\begin{aligned}
&\left(\frac{1}{\xi}\frac{\partial^2}{\partial t^2} + p^2\right) F_{\mathrm{S}}^{(\mathrm{g})}(t,t';p) + \left(1 - \frac{1}{\xi}\right) p \, \frac{\partial}{\partial t} \widetilde{F}_{\mathrm{V}_1}^{(\mathrm{g})}(t,t';p) \\
&\qquad = \int_{t_0}^{t} \mathrm{d}t'' \left[ \Pi_{(\rho)\mathrm{S}}(t,t'';p)\, F_{\mathrm{S}}^{(\mathrm{g})}(t'',t';p) + \widetilde{\Pi}_{(\rho)\mathrm{V}_2}(t,t'';p)\, \widetilde{F}_{\mathrm{V}_1}^{(\mathrm{g})}(t'',t';p) \right] \\
&\qquad\quad - \int_{t_0}^{t'} \mathrm{d}t'' \left[ \Pi_{(F)\mathrm{S}}(t,t'';p)\, \rho_{\mathrm{S}}^{(\mathrm{g})}(t'',t';p) + \widetilde{\Pi}_{(F)\mathrm{V}_2}(t,t'';p)\, \widetilde{\rho}_{\mathrm{V}_1}^{(\mathrm{g})}(t'',t';p) \right], \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.49\mathrm{a})
\end{aligned}$$

$$\left(\frac{\partial^2}{\partial t^2}+\frac{p^2}{\xi}\right)\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';p)+\left(1-\frac{1}{\xi}\right)p\,\frac{\partial}{\partial t}F^{(\mathrm{g})}_{\mathrm{S}}(t,t';p)$$
$$=\int_{t_0}^{t}\mathrm{d}t''\left[\widetilde{\Pi}_{(\rho)\mathrm{V}_1}(t,t'';p)\,F^{(\mathrm{g})}_{\mathrm{S}}(t'',t';p)+\Pi_{(\rho)\mathrm{L}}(t,t'';p)\,\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(t'',t';p)\right]$$
$$-\int_{t_0}^{t'}\mathrm{d}t''\left[\widetilde{\Pi}_{(F)\mathrm{V}_1}(t,t'';p)\,\rho^{(\mathrm{g})}_{\mathrm{S}}(t'',t';p)+\Pi_{(F)\mathrm{L}}(t,t'';p)\,\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(t'',t';p)\right],\tag{3.49b}$$

$$\left(\frac{1}{\xi}\frac{\partial^2}{\partial t^2}+p^2\right)\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';p)+\left(1-\frac{1}{\xi}\right)p\,\frac{\partial}{\partial t}F^{(\mathrm{g})}_{\mathrm{L}}(t,t';p)$$
$$=\int_{t_0}^{t}\mathrm{d}t''\left[\Pi_{(\rho)\mathrm{S}}(t,t'';p)\,\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t'',t';p)+\widetilde{\Pi}_{(\rho)\mathrm{V}_2}(t,t'';p)\,F^{(\mathrm{g})}_{\mathrm{L}}(t'',t';p)\right]$$
$$-\int_{t_0}^{t'}\mathrm{d}t''\left[\Pi_{(F)\mathrm{S}}(t,t'';p)\,\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(t'',t';p)+\widetilde{\Pi}_{(F)\mathrm{V}_2}(t,t'';p)\,\rho^{(\mathrm{g})}_{\mathrm{L}}(t'',t';p)\right],\tag{3.49c}$$

$$\left(\frac{\partial^2}{\partial t^2}+\frac{p^2}{\xi}\right)F^{(\mathrm{g})}_{\mathrm{L}}(t,t';p)+\left(1-\frac{1}{\xi}\right)p\,\frac{\partial}{\partial t}\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';p)$$
$$=\int_{t_0}^{t}\mathrm{d}t''\left[\widetilde{\Pi}_{(\rho)\mathrm{V}_1}(t,t'';p)\,\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t'',t';p)+\Pi_{(\rho)\mathrm{L}}(t,t'';p)\,F^{(\mathrm{g})}_{\mathrm{L}}(t'',t';p)\right]$$
$$-\int_{t_0}^{t'}\mathrm{d}t''\left[\widetilde{\Pi}_{(F)\mathrm{V}_1}(t,t'';p)\,\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(t'',t';p)+\Pi_{(F)\mathrm{L}}(t,t'';p)\,\rho^{(\mathrm{g})}_{\mathrm{L}}(t'',t';p)\right],\tag{3.49d}$$

$$\left(\frac{\partial^2}{\partial t^2}+p^2\right)F^{(\mathrm{g})}_{\mathrm{T}}(t,t';p)$$
$$=\int_{t_0}^{t}\mathrm{d}t''\,\Pi_{(\rho)\mathrm{T}}(t,t'';p)\,F^{(\mathrm{g})}_{\mathrm{T}}(t'',t';p)-\int_{t_0}^{t'}\mathrm{d}t''\,\Pi_{(F)\mathrm{T}}(t,t'';p)\,\rho^{(\mathrm{g})}_{\mathrm{T}}(t'',t';p)\,.\tag{3.49e}$$

**Fermion Equations of Motion** The EoMs for the fermion spectral function read:

$$\frac{\partial}{\partial t}\rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';p)=\;m^{(\mathrm{f})}\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';p)+p\,\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';p)$$
$$+\int_{t'}^{t}\mathrm{d}t''\Big[\;\Sigma_{(\rho)\mathrm{S}}(t,t'';p)\,\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t'',t';p)+\widetilde{\Sigma}^{0}_{(\rho)\mathrm{V}}(t,t'';p)\,\rho^{(\mathrm{f})}_{\mathrm{S}}(t'',t';p)$$
$$-\Sigma_{(\rho)\mathrm{V}}(t,t'';p)\,\rho^{(\mathrm{f})}_{\mathrm{T}}(t'',t';p)+\Sigma_{(\rho)\mathrm{T}}(t,t'';p)\,\rho^{(\mathrm{f})}_{\mathrm{V}}(t'',t';p)\Big],\tag{3.50a}$$

$$\frac{\partial}{\partial t}\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';p)=-m^{(\mathrm{f})}\rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';p)-p\,\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';p)$$
$$-\int_{t'}^{t}\mathrm{d}t''\Big[\;\Sigma_{(\rho)\mathrm{S}}(t,t'';p)\rho^{(\mathrm{f})}_{\mathrm{S}}(t'',t';p)-\widetilde{\Sigma}^{0}_{(\rho)\mathrm{V}}(t,t'';p)\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t'',t';p)$$

$$- \Sigma_{(\rho)\mathrm{V}}(t,t'';p)\rho^{(\mathrm{f})}_{\mathrm{V}}(t'',t';p) - \Sigma_{(\rho)\mathrm{T}}(t,t'';p)\rho^{(\mathrm{f})}_{\mathrm{T}}(t'',t';p)\Big]\,, \tag{3.50b}$$

$$\begin{aligned}\frac{\partial}{\partial t}\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';p) = {} & p\,\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';p) - m^{(\mathrm{f})}\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';p) \\ & - \int_{t'}^{t}\mathrm{d}t''\Big[\ \Sigma_{(\rho)\mathrm{S}}(t,t'';p)\,\rho^{(\mathrm{f})}_{\mathrm{T}}(t'',t';p) + \Sigma_{(\rho)\mathrm{T}}(t,t'';p)\,\rho^{(\mathrm{f})}_{\mathrm{S}}(t'',t';p) \\ & \qquad + \Sigma_{(\rho)\mathrm{V}}(t,t'';p)\,\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t'',t';p) - \widetilde{\Sigma}^{0}_{(\rho)\mathrm{V}}(t,t'';p)\rho^{(\mathrm{f})}_{\mathrm{V}}(t'',t';p)\Big]\,,\end{aligned} \tag{3.50c}$$

$$\begin{aligned}\frac{\partial}{\partial t}\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';p) = {} & -p\,\rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';p) + m^{(\mathrm{f})}\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';p) \\ & + \int_{t'}^{t}\mathrm{d}t''\Big[\ \Sigma_{(\rho)\mathrm{S}}(t,t'';p)\,\rho^{(\mathrm{f})}_{\mathrm{V}}(t'',t';p) + \Sigma_{(\rho)\mathrm{V}}(t,t'';p)\,\rho^{(\mathrm{f})}_{\mathrm{S}}(t'',t';p) \\ & \qquad + \widetilde{\Sigma}^{0}_{(\rho)\mathrm{V}}(t,t'';p)\,\rho^{(\mathrm{f})}_{\mathrm{T}}(t'',t';p) - \Sigma_{(\rho)\mathrm{T}}(t,t'';p)\,\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t'',t';p)\Big]\,,\end{aligned} \tag{3.50d}$$

and the EoMs for the statistical function read:

$$\begin{aligned}\frac{\partial}{\partial t}F^{(\mathrm{f})}_{\mathrm{S}}(t,t';p) = {} & m^{(\mathrm{f})}\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';p) + p\,F^{(\mathrm{f})}_{\mathrm{T}}(t,t';p) \\ & + \int_{t_0}^{t}\mathrm{d}t''\Big[\ \Sigma_{(\rho)\mathrm{S}}(t,t'';p)\,\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t'',t';p) + \widetilde{\Sigma}^{0}_{(\rho)\mathrm{V}}(t,t'';p)\,F^{(\mathrm{f})}_{\mathrm{S}}(t'',t';p) \\ & \qquad - \Sigma_{(\rho)\mathrm{V}}(t,t'';p)\,F^{(\mathrm{f})}_{\mathrm{T}}(t'',t';p) + \Sigma_{(\rho)\mathrm{T}}(t,t'';p)\,F^{(\mathrm{f})}_{\mathrm{V}}(t'',t';p)\Big] \\ & - \int_{t_0}^{t'}\mathrm{d}t''\Big[\ \Sigma_{(F)\mathrm{S}}(t,t'';p)\,\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t'',t';p) + \widetilde{\Sigma}^{0}_{(F)\mathrm{V}}(t,t'';p)\,\rho^{(\mathrm{f})}_{\mathrm{S}}(t'',t';p) \\ & \qquad - \Sigma_{(F)\mathrm{V}}(t,t'';p)\,\rho^{(\mathrm{f})}_{\mathrm{T}}(t'',t';p) + \Sigma_{(F)\mathrm{T}}(t,t'';p)\,\rho^{(\mathrm{f})}_{\mathrm{V}}(t'',t';p)\Big]\,,\end{aligned} \tag{3.51a}$$

$$\begin{aligned}\frac{\partial}{\partial t}\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';p) = {} & -m^{(\mathrm{f})}F^{(\mathrm{f})}_{\mathrm{S}}(t,t';p) - p\,F^{(\mathrm{f})}_{\mathrm{V}}(t,t';p) \\ & - \int_{t_0}^{t}\mathrm{d}t''\Big[\ \Sigma_{(\rho)\mathrm{S}}(t,t'';p)F^{(\mathrm{f})}_{\mathrm{S}}(t'',t';p) - \widetilde{\Sigma}^{0}_{(\rho)\mathrm{V}}(t,t'';p)\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t'',t';p) \\ & \qquad - \Sigma_{(\rho)\mathrm{V}}(t,t'';p)F^{(\mathrm{f})}_{\mathrm{V}}(t'',t';p) - \Sigma_{(\rho)\mathrm{T}}(t,t'';p)F^{(\mathrm{f})}_{\mathrm{T}}(t'',t';p)\Big] \\ & + \int_{t_0}^{t'}\mathrm{d}t''\Big[\ \Sigma_{(F)\mathrm{S}}(t,t'';p)\rho^{(\mathrm{f})}_{\mathrm{S}}(t'',t';p) - \widetilde{\Sigma}^{0}_{(F)\mathrm{V}}(t,t'';p)\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t'',t';p) \\ & \qquad - \Sigma_{(F)\mathrm{V}}(t,t'';p)\rho^{(\mathrm{f})}_{\mathrm{V}}(t'',t';p) - \Sigma_{(F)\mathrm{T}}(t,t'';p)\rho^{(\mathrm{f})}_{\mathrm{T}}(t'',t';p)\Big]\,,\end{aligned} \tag{3.51b}$$

$$\frac{\partial}{\partial t}F^{(\mathrm{f})}_{\mathrm{V}}(t,t';p) = \ p\,\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';p) - m^{(\mathrm{f})}F^{(\mathrm{f})}_{\mathrm{T}}(t,t';p)$$

$$
\begin{aligned}
&- \int_{t_0}^{t} \mathrm{d}t'' \Big[ \; \Sigma_{(\rho)\mathrm{S}}(t,t'';p)\, F^{(\mathrm{f})}_{\mathrm{T}}(t'',t';p) + \Sigma_{(\rho)\mathrm{T}}(t,t'';p)\, F^{(\mathrm{f})}_{\mathrm{S}}(t'',t';p) \\
&\qquad + \Sigma_{(\rho)\mathrm{V}}(t,t'';p)\, \widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t'',t';p) - \widetilde{\Sigma}^{0}_{(\rho)\mathrm{V}}(t,t'';p) F^{(\mathrm{f})}_{\mathrm{V}}(t'',t';p) \Big] \\
&+ \int_{t_0}^{t'} \mathrm{d}t'' \Big[ \; \Sigma_{(F)\mathrm{S}}(t,t'';p)\, \rho^{(\mathrm{f})}_{\mathrm{T}}(t'',t';p) + \Sigma_{(F)\mathrm{T}}(t,t'';p)\, \rho^{(\mathrm{f})}_{\mathrm{S}}(t'',t';p) \\
&\qquad + \Sigma_{(F)\mathrm{V}}(t,t'';p)\, \widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t'',t';p) - \widetilde{\Sigma}^{0}_{(F)\mathrm{V}}(t,t'';p) \rho^{(\mathrm{f})}_{\mathrm{V}}(t'',t';p) \Big] \,,
\end{aligned}
\tag{3.51c}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t} F^{(\mathrm{f})}_{\mathrm{T}}(t,t';p) &= -p\, F^{(\mathrm{f})}_{\mathrm{S}}(t,t';p) + m^{(\mathrm{f})} F^{(\mathrm{f})}_{\mathrm{V}}(t,t';p) \\
&+ \int_{t_0}^{t} \mathrm{d}t'' \Big[ \; \Sigma_{(\rho)\mathrm{S}}(t,t'';p)\, F^{(\mathrm{f})}_{\mathrm{V}}(t'',t';p) + \Sigma_{(\rho)\mathrm{V}}(t,t'';p)\, F^{(\mathrm{f})}_{\mathrm{S}}(t'',t';p) \\
&\qquad + \widetilde{\Sigma}^{0}_{(\rho)\mathrm{V}}(t,t'';p)\, F^{(\mathrm{f})}_{\mathrm{T}}(t'',t';p) - \Sigma_{(\rho)\mathrm{T}}(t,t'';p)\, \widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t'',t';p) \Big] \\
&- \int_{t_0}^{t'} \mathrm{d}t'' \Big[ \; \Sigma_{(F)\mathrm{S}}(t,t'';p)\, \rho^{(\mathrm{f})}_{\mathrm{V}}(t'',t';p) + \Sigma_{(F)\mathrm{V}}(t,t'';p)\, \rho^{(\mathrm{f})}_{\mathrm{S}}(t'',t';p) \\
&\qquad + \widetilde{\Sigma}^{0}_{(F)\mathrm{V}}(t,t'';p)\, \rho^{(\mathrm{f})}_{\mathrm{T}}(t'',t';p) - \Sigma_{(F)\mathrm{T}}(t,t'';p)\, \widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t'',t';p) \Big] \,.
\end{aligned}
\tag{3.51d}
$$

These are the EoMs one has to solve. Note, however, that the photon EoMs are very inconvenient to work with for two reasons: First, it is not obvious that the EoMs are well-defined for Landau gauge (corresponding to the limit $\xi \to 0$) due to the appearance of various factors of $1/\xi$. And second, in addition to second derivatives with respect to time, there appear also first derivatives. This is very impractical at least from a numerical point of view. It is therefore desirable to find a different formulation for the photon EoMs which does not exhibit these disadvantages.

## 3.3 Reformulation of the Photon Equations of Motion

We will now introduce a reformulation of the EoMs which is of little practical use if one is interested in solving the full EoMs numerically, but which is very instructive for learning about an interesting feature of the EoMs which we will discuss in great detail later on.

We start by introducing an auxiliary field, the so-called *Nakanishi–Lautrup (*NL*) field $B$* [Nak66, Lau66] which is commonly used in the path integral quantization of gauge theories in order to close the BRST algebra. Only the gauge part of the classical action is modified, and the action involving the auxiliary field now reads:

$$
S_{\mathrm{NL}}[A,B] = \int_x \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + B\, \partial^\mu A_\mu + \frac{\xi}{2} B^2 \right) = S_{\mathrm{g}}[A] + S_B[A,B] \tag{3.52}
$$

with

$$S_B[A,B] = \int_x \left(B\,\partial^\mu A_\mu + \frac{\xi}{2}\,B^2\right) = \frac{1}{2\xi}\int_x \Big[(\partial^\mu A_\mu + \xi\,B)^2 - (\partial^\mu A_\mu)^2\Big]$$
$$= S_{\rm gf}[A] + \frac{1}{2\xi}\int_x (\partial^\mu A_\mu + \xi\,B)^2\,, \tag{3.53}$$

where we have completed the square in the second line.[34] Note that there is no kinetic term for $B$, and since it appears only quadratically in the action, it can easily be integrated out in the path integral.

It is now convenient to introduce the composite field $(\widetilde{A}_m) = (A_\mu, B)$ (with $m = 0, \ldots, 4$, so that $\widetilde{A}_\mu = A_\mu$ and $\widetilde{A}_4 = B$). Then the action (3.52) can be written equivalently as

$$S_{\rm NL}[\widetilde{A}] = \frac{\rm i}{2}\int_{x,y} \widetilde{A}_m(x)\Big(\widetilde{D_0^{-1}}\Big)^{mn}(x,y)\widetilde{A}_n(y) \tag{3.54}$$

with the inverse free propagator

$${\rm i}\Big(\widetilde{D_0^{-1}}\Big)^{mn}(x,y) = \frac{\delta^2 S_{\rm NL}[\widetilde{A}]}{\delta\widetilde{A}_m(x)\delta\widetilde{A}_n(y)}\,, \tag{3.55}$$

so that

$$\begin{aligned}
{\rm i}\Big(\widetilde{D_0^{-1}}\Big)^{\mu\nu}(x,y) &= \Big(g^{\mu\nu}\Box_x - \partial_x^\mu\partial_x^\nu\Big)\delta^4(x-y)\,,\\
{\rm i}\Big(\widetilde{D_0^{-1}}\Big)^{\mu 4}(x,y) &= -\partial_x^\mu\delta^4(x-y)\,,\\
{\rm i}\Big(\widetilde{D_0^{-1}}\Big)^{4\mu}(x,y) &= \partial_x^\mu\delta^4(x-y)\,,\\
{\rm i}\Big(\widetilde{D_0^{-1}}\Big)^{44}(x,y) &= \xi\,\delta^4(x-y)
\end{aligned} \tag{3.56}$$

or

$${\rm i}\Big(\widetilde{D_0^{-1}}\Big)^{mn}(x,y) = \Big[\delta_\mu^m\delta_\nu^n(g^{\mu\nu}\Box_x - \partial_x^\mu\partial_x^\nu) + (\delta_4^m\delta_\mu^n - \delta_\mu^m\delta_4^n)\partial_x^\mu + \delta_4^m\delta_4^n\,\xi\Big]\delta^4(x-y)\,. \tag{3.57}$$

Since the free inverse propagators are translation invariant, we can Fourier transform them

[34] If we quantize the theory by establishing a path integral, the introduction of the NL field can be interpreted as a Hubbard–Stratonovich transformation since

$$\exp\left(-\frac{1}{2\xi}\,(\partial^\mu A_\mu)^2\right) = \mathcal{N}\int\mathcal{D}B\,\exp\left(\frac{\xi}{2}\,B^2 + B\,\partial^\mu A_\mu\right)\,,$$

so that

$$\mathcal{N}\int\mathcal{D}B\;{\rm e}^{{\rm i}\,S_B[A,B]} = {\rm e}^{{\rm i}\,S_{\rm gf}[A]}\,.$$

to obtain

$$\begin{aligned}
\left(\widetilde{D}_0^{-1}\right)^{\mu\nu}(p) &= \mathrm{i}(g^{\mu\nu}p^2 - p^\mu p^\nu)\,,\\
\left(\widetilde{D}_0^{-1}\right)^{\mu 4}(p) &= p^\mu\,,\\
\left(\widetilde{D}_0^{-1}\right)^{4\mu}(p) &= -p^\mu\,,\\
\left(\widetilde{D}_0^{-1}\right)^{44}(p) &= -\,\mathrm{i}\,\xi\,.
\end{aligned} \tag{3.58}$$

Employing the formula[35]

$$\det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det(D)\det(A - BD^{-1}C)$$

for the determinant of a block matrix, we obtain:

$$\begin{aligned}
&\det\Big(\mathrm{i}(\widetilde{D}_0^{-1})^{mn}(p)\Big)\\
&\quad = -\det\Big((\widetilde{D}_0^{-1})^{\mu\nu}(p) - (\widetilde{D}_0^{-1})^{\mu 4}(p)(\widetilde{D}_0^{-1})^{44}(p)^{-1}(\widetilde{D}_0^{-1})^{4\nu}(p)\Big)\det\Big((\widetilde{D}_0^{-1})^{44}(p)\Big)\\
&\quad = -\xi\det\left(g^{\mu\nu}p^2 - \left(1 - \frac{1}{\xi}\right)p^\mu p^\nu\right)\\
&\quad = p^2\,,
\end{aligned} \tag{3.59}$$

where in the last step we have employed the matrix determinant lemma. Note that the determinant of $\mathrm{i}\,\widetilde{D}_0^{-1}$ is nonvanishing, so that $\widetilde{D}_0^{-1}$ is invertible (in contrast to $D_0^{-1}$), and independent of $\xi$. Therefore, the free propagator exists for each choice of gauge. We find:

$$\begin{aligned}
\widetilde{D}_{0\mu\nu}(p) &= D_{0\mu\nu}(p) = -\frac{\mathrm{i}}{p^2}\left[g_{\mu\nu} - (1-\xi)\,\frac{p_\mu p_\nu}{p^2}\right]\,,\\
\widetilde{D}_{0\mu 4}(p) &= D_\mu^{(AB)}(p) = -\frac{p_\mu}{p^2}\,,\\
\widetilde{D}_{04\mu}(p) &= D_{0\,\mu}^{(BA)}(p) = \frac{p_\mu}{p^2}\,,\\
\widetilde{D}_{044}(p) &= D_0^{(BB)}(p) = 0\,.
\end{aligned} \tag{3.60}$$

Defining

$$\widetilde{D}_{mn}(x,y) = \langle \mathrm{T}\widetilde{A}_m(x)\widetilde{A}_n(y)\rangle\,, \tag{3.61}$$

we have

$$\begin{aligned}
\widetilde{D}_{\mu\nu}(x,y) &= \langle \mathrm{T}\widetilde{A}_\mu(x)\widetilde{A}_\nu(y)\rangle = \langle \mathrm{T}A_\mu(x)A_\nu(y)\rangle = D_{\mu\nu}^{(AA)}(x,y) = D_{\mu\nu}(x,y)\,,\\
\widetilde{D}_{\mu 4}(x,y) &= \langle \mathrm{T}\widetilde{A}_\mu(x)\widetilde{A}_4(y)\rangle = \langle \mathrm{T}A_\mu(x)B(y)\rangle = D_\mu^{(AB)}(x,y)\,,\\
\widetilde{D}_{4\mu}(x,y) &= \langle \mathrm{T}\widetilde{A}_4(x)\widetilde{A}_\mu(y)\rangle = \langle \mathrm{T}B(x)A_\mu(y)\rangle = D_\mu^{(BA)}(x,y)\,,\\
\widetilde{D}_{44}(x,y) &= \langle \mathrm{T}\widetilde{A}_4(x)\widetilde{A}_4(y)\rangle = \langle \mathrm{T}B(x)B(y)\rangle = D^{(BB)}(x,y)\,,
\end{aligned} \tag{3.62}$$

[35] Since $D$ is just a number (i. e. a one-by-one matrix), the formula can further be simplified to $\det(AD - BC)$.

or

$$\widetilde{D}_{mn}(x,y) = \delta^\mu_m \delta^\nu_n D_{\mu\nu}(x,y) + \delta^\mu_m \delta^4_n D^{(AB)}_\mu(x,y) + \delta^4_m \delta^\mu_n D^{(BA)}_\mu(x,y) + \delta^4_m \delta^4_n D^{(BB)}(x,y)\,. \quad (3.63)$$

The first correlation function is just the usual photon propagator, while the others contain at least one NL field operator. The 2PI effective action then reads:[36]

$$\Gamma_{2\mathrm{PI}}[\widetilde{D},S] = \frac{\mathrm{i}}{2}\operatorname{Tr}\ln \widetilde{D}^{-1} + \frac{\mathrm{i}}{2}\operatorname{Tr}\Big(\widetilde{D}_0^{-1}\widetilde{D}\Big) - \mathrm{i}\operatorname{Tr}\ln S^{-1} - \mathrm{i}\operatorname{Tr}\Big(S_0^{-1}S\Big) + \Gamma_2[S,\widetilde{D}]\,. \quad (3.64)$$

Here we have assumed that the expectation values of the photon and fermion fields vanish. We further assume that the expectation value of the NL field vanishes; in fact, in the operator formalism this is a consequence of a certain physicality condition (see App. A).

From the stationarity condition of the photon part,

$$\frac{\delta\Gamma_{2\mathrm{PI}}[\widetilde{D},S]}{\delta\widetilde{D}_{mn}(x,y)} = 0\,, \quad (3.65)$$

it follows that the EOM is given by:

$$\Big(\widetilde{D}^{-1}\Big)^{mn}(x,y) = \Big(\widetilde{D}_0^{-1}\Big)^{mn}(x,y) - \widetilde{\Pi}^{mn}(x,y) \quad (3.66)$$

with the self-energy

$$\widetilde{\Pi}^{mn}(x,y) = 2\,\mathrm{i}\,\frac{\delta\Gamma_{2\mathrm{PI}}[\widetilde{D},S]}{\delta\widetilde{D}_{mn}(x,y)}\,. \quad (3.67)$$

Now, the important observation is that the 2PI part of the 2PI effective action does not depend on any correlators involving the NL field, i.e. instead of on the full $\widetilde{D}$, it only depends on the pure photon correlator $D$. The reason is simple: The only way to include a correlator involving an NL field into a given diagram is to replace a photon line by a photon line which is connected to a mixed correlator; schematically

$$\begin{aligned} D = D^{(AA)} \to \quad & D^{(AA)} D^{(AB)} \mathcal{M}(D^{(AA)}, D^{(AB)}, D^{(BA)}, D^{(BB)}, S) D^{(BA)} D^{(AA)} \\ & = D D^{(AB)} \mathcal{M}(\widetilde{D}, S) D^{(BA)} D\,, \end{aligned}$$

where $\mathcal{M}$ is a Lorentz scalar potentially depending on all possible propagators. Such a diagram, however, can never be 2PI, since cutting the two photon lines leaves us with a diagram $D^{(AB)}\mathcal{M}(\widetilde{D},S)D^{(BA)}$ which is disconnected from the rest. We will see a concrete example in Sec. 6.1 when we consider a finite truncation of the 2PI effective action.

[36] Note that the matrices consisting of the $AA$-, $AB$-, $BA$- and $BB$-components are not block diagonal, so that the terms in the one-loop part of the 2PI effective action do not factorize.

Therefore, $\Gamma_{2\mathrm{PI}}$ may only depend on $D$, not on (all components of) $\widetilde{D}$. Then, in terms of its components, we have

$$
\begin{aligned}
\widetilde{\Pi}^{\mu\nu}(x,y) &= 2\,\mathrm{i}\,\frac{\delta\Gamma_{2\mathrm{PI}}[\widetilde{D},S]}{\delta\widetilde{D}_{\mu\nu}(x,y)} = 2\,\mathrm{i}\,\frac{\delta\Gamma_{2\mathrm{PI}}[\widetilde{D},S]}{\delta D_{\mu\nu}(x,y)} = \Pi^{\mu\nu}(x,y)\,,\\
\widetilde{\Pi}^{\mu4}(x,y) &= 2\,\mathrm{i}\,\frac{\delta\Gamma_{2\mathrm{PI}}[\widetilde{D},S]}{\delta\widetilde{D}_{\mu4}(x,y)} = 2\,\mathrm{i}\,\frac{\delta\Gamma_{2\mathrm{PI}}[\widetilde{D},S]}{\delta D^{(AB)}_{\mu}(x,y)} = \Pi^{(AB)\mu}(x,y) = 0\,,\\
\widetilde{\Pi}^{4\mu}(x,y) &= 2\,\mathrm{i}\,\frac{\delta\Gamma_{2\mathrm{PI}}[\widetilde{D},S]}{\delta\widetilde{D}_{4\mu}(x,y)} = 2\,\mathrm{i}\,\frac{\delta\Gamma_{2\mathrm{PI}}[\widetilde{D},S]}{\delta D^{(BA)}_{\mu}(x,y)} = \Pi^{(BA)\mu}(x,y) = 0\,,\\
\widetilde{\Pi}^{44}(x,y) &= 2\,\mathrm{i}\,\frac{\delta\Gamma_{2\mathrm{PI}}[\widetilde{D},S]}{\delta\widetilde{D}_{44}(x,y)} = 2\,\mathrm{i}\,\frac{\delta\Gamma_{2\mathrm{PI}}[\widetilde{D},S]}{\delta D^{(BB)}(x,y)} = \Pi^{(BB)}(x,y) = 0\,,
\end{aligned}
\tag{3.68}
$$

or

$$
\widetilde{\Pi}^{mn}(x,y) = \delta^m_\mu \delta^n_\nu \Pi^{\mu\nu}(x,y)\,. \tag{3.69}
$$

The EoM (3.66) can then be rewritten as

$$
\int_z \left(\widetilde{D_0^{-1}}\right)^{mk}(x,z)\widetilde{D}_{kn}(z,y) = \delta^m_n \delta^4(x-y) + \int_z \widetilde{\Pi}^{mk}(x,z)\widetilde{D}_{kn}(z,y)\,, \tag{3.70}
$$

so that

$$
\int_z \left[\left(\widetilde{D_0^{-1}}\right)^{\mu\lambda}(x,z)\widetilde{D}_{\lambda\nu}(z,y) + \left(\widetilde{D}_0{}^{-1}\right)^{\mu4}(x,z)\widetilde{D}_{4\nu}(z,y)\right] = \delta^\mu_\nu \delta^4(x-y) + \int_z \widetilde{\Pi}^{\mu\lambda}(x,z)\widetilde{D}_{\lambda\nu}(z,y)\,, \tag{3.71a}
$$

$$
\int_z \left[\left(\widetilde{D_0^{-1}}\right)^{\mu\nu}(x,z)\widetilde{D}_{\nu4}(z,y) + \left(\widetilde{D}_0{}^{-1}\right)^{\mu4}(x,z)\widetilde{D}_{44}(z,y)\right] = \int_z \widetilde{\Pi}^{\mu\nu}(x,z)\widetilde{D}_{\nu4}(z,y)\,, \tag{3.71b}
$$

$$
\int_z \left[\left(\widetilde{D}_0{}^{-1}\right)^{4\mu}(x,z)\widetilde{D}_{\mu\nu}(z,y) + \left(\widetilde{D}_0{}^{-1}\right)^{44}(x,z)\widetilde{D}_{4\nu}(z,y)\right] = 0\,, \tag{3.71c}
$$

$$
\int_z \left[\left(\widetilde{D}_0{}^{-1}\right)^{4\mu}(x,z)\widetilde{D}_{\mu4}(z,y) + \left(\widetilde{D}_0{}^{-1}\right)^{44}(x,z)\widetilde{D}_{44}(z,y)\right] = \delta^4(x-y)\,. \tag{3.71d}
$$

With the free inverse propagators (3.56), we then have:

$$
\left(g^{\mu\lambda}\Box_x - \partial^\mu_x\partial^\lambda_x\right)D_{\lambda\nu}(x,y) - \partial^\mu_x D^{(BA)}_\nu(x,y) = \mathrm{i}\,\delta^\mu_\nu\delta^4(x-y) + \mathrm{i}\int_z \Pi^{\mu\lambda}(x,z)D_{\lambda\nu}(z,y)\,, \tag{3.72a}
$$

$$
\left(g^{\mu\nu}\Box_x - \partial^\mu_x\partial^\nu_x\right)D^{(AB)}_\nu(x,y) - \partial^\mu_x D^{(BB)}(x,y) = \mathrm{i}\int_z \Pi^{\mu\nu}(x,z)D^{(AB)}_\nu(z,y)\,, \tag{3.72b}
$$

$$
\partial^\mu_x D_{\mu\nu}(x,y) + \xi\, D^{(BA)}_\nu(x,y) = 0\,, \tag{3.72c}
$$

$$
\partial^\mu_x D^{(AB)}_\mu(x,y) + \xi\, D^{(BB)}(x,y) = \mathrm{i}\,\delta^4(x-y)\,. \tag{3.72d}
$$

Plugging (3.72c) into (3.72a), we obtain:

$$
\Box_x D_{\mu\nu}(x,y) - (1-\xi)\,\partial_{x\mu}D^{(BA)}_\nu(x,y) = \mathrm{i}\,g_{\mu\nu}\delta^4(x-y) + \mathrm{i}\int_z \Pi_\mu{}^\lambda(x,z)D_{\lambda\nu}(z,y)\,, \tag{3.73}
$$

Further, applying $\partial_{x\mu}$ to (3.72a), we have:[37]

$$\Box_x D_\nu^{(BA)}(x,y) = -\,\mathrm{i}\,\partial_{x\nu}\delta^4(x-y) - \mathrm{i}\int_z \partial_{x\mu}\Pi^{\mu\lambda}(x,z)D_{\lambda\nu}(z,y)\,. \tag{3.74}$$

Decomposing these equations in their spectral and statistical components, we obtain:

$$\Box_x\rho^{(\mathrm{g})}_{\mu\nu}(x,y) - (1-\xi)\,\partial_{x\mu}\rho_\nu^{(BA)}(x,y) = \int_{y^0}^{x^0}\mathrm{d}z\,\Pi_{(\rho)\mu}{}^{\lambda}(x,z)\rho^{(\mathrm{g})}_{\lambda\nu}(z,y)\,, \tag{3.75a}$$

$$\begin{aligned}\Box_x F^{(\mathrm{g})}_{\mu\nu}(x,y) - (1-\xi)\,\partial_{x\mu}F_\nu^{(BA)}(x,y) = &\int_0^{x^0}\mathrm{d}z\,\Pi_{(\rho)\mu}{}^{\lambda}(x,z)F^{(\mathrm{g})}_{\lambda\nu}(z,y)\\ &-\int_0^{y^0}\mathrm{d}z\,\Pi_{(F)\mu}{}^{\lambda}(x,z)\rho^{(\mathrm{g})}_{\lambda\nu}(z,y)\,,\end{aligned} \tag{3.75b}$$

and

$$\Box_x\rho_\nu^{(BA)}(x,y) = -\int_{y^0}^{x^0}\mathrm{d}z\,\partial_{x\mu}\Pi^{\mu\lambda}_{(\rho)}(x,z)\rho^{(\mathrm{g})}_{\lambda\nu}(z,y)\,, \tag{3.76a}$$

$$\Box_x F_\nu^{(BA)}(x,y) = -\int_0^{x^0}\mathrm{d}z\,\partial_{x\mu}\Pi^{\mu\lambda}_{(\rho)}(x,z)F^{(\mathrm{g})}_{\lambda\nu}(z,y) + \int_0^{y^0}\mathrm{d}z\,\partial_{x\mu}\Pi^{\mu\lambda}_{(F)}(x,z)\rho^{(\mathrm{g})}_{\lambda\nu}(z,y)\,. \tag{3.76b}$$

On first sight, it looks as if the EoMs for $\rho_\mu^{(BA)}$ and $F_\mu^{(BA)}$ are free, since only the longitudinal part $\partial_{x\mu}\Pi^{\mu\nu}(x,y)$ of the photon self-energy enters the memory integrals, and that is guaranteed to vanish by the Ward identities. If this were the case, one could easily solve

[37] Similarly, plugging (3.72d) into (3.72b), we obtain:

$$\Box_x D_\mu^{(AB)}(x,y) - (1-\xi)\,\partial_{x\mu}D^{(BB)}(x,y) = \mathrm{i}\,\partial_{x\mu}\delta^4(x-y) + \mathrm{i}\int_z \Pi_\mu{}^{\nu}(x,z)D_\nu^{(AB)}(z,y)\,,$$

while applying $\partial_{x\mu}$ to (3.72b), we have:

$$\Box_x D^{(BB)}(x,y) = -\,\mathrm{i}\int_z \partial_{x\mu}\Pi^{\mu\nu}(x,z)D_\nu^{(AB)}(z,y)\,.$$

Various other identities can be derived as well. For instance, combining Eqs. (3.72c) and (3.72d), we obtain

$$\partial_x^\mu\partial_y^\nu D_{\mu\nu}(x,y) + \xi\left[\mathrm{i}\,\delta^4(x-y) - \xi\,D^{(BB)}(x,y)\right] = 0\,.$$

In the exact theory, where $D^{(BB)}$ satisfies a free EoM, which, together with its initial conditions, implies that it vanishes identically, this identity reduces to

$$\partial_x^\mu\partial_y^\nu D_{\mu\nu}(x,y) = -\,\mathrm{i}\,\xi\,\delta^4(x-y)\,,$$

which is a well-known expression of the Ward identity stating that the longitudinal part of the photon propagator does not get modified by quantum fluctuations.

Finally note that from Eq. (3.72c), it follows that $\partial_x^\mu\partial_x^\lambda D_{\lambda\nu}(x,y) + \xi\,\partial_x^\mu D_\nu^{(BA)}(x,y) = 0$, so that

$$\partial_x^\mu D_\nu^{(BA)}(x,y) = -\frac{1}{\xi}\,\partial_x^\mu\partial_x^\lambda D_{\lambda\nu}(x,y)\,.$$

Plugging this into Eq. (3.72a), we get back the original EoMs.

their EoMs analytically (given their initial conditions which will be provided in Sec. 3.4) and plug the results back into the photon EoMs (3.75), thereby getting rid of any reference to the auxiliary field correlators altogether.

For a finitely truncated 2PI effective action, however, it turns out that photon self-energies as defined previously are *not* purely transverse, see Chap. 5.[38] Therefore, the EoMs for the auxiliary correlators cannot easily be solved analytically, and instead of 32 EoMs, we now have 40 EoMs. As already stated at the beginning of this section, however, the aim of this reformulation of the photon EoMs is not to make the equations more tractable from a practical point of view, but to understand their structure better. After we have provided the initial conditions needed to solve the EoMs in the next section, we will solve the free photon EoMs (for which the photon self-energy vanishes identically so that the EoMs of the auxiliary field correlators are in fact free) by employing the reformulation presented in this section.

## 3.4 Initial Conditions

In this section, we will provide the initial conditions for the photon and fermion two-point functions which are necessary to solve their respective EoMs. Before we do so, however, we will briefly elaborate on the connection between (Gaussian) density operators—or rather the distribution functions they imply—and initial conditions for (one- and two-point) correlation functions.

### 3.4.1 Gaussian Distribution Functions

#### Finite Dimensional Gaussian Distributions

It is instructive to first consider finite dimensional Gaussian distribution functions, and in particular the case of one and two random variables, respectively. In fact, these two cases will make it easy to establish the connection to a Gaussian density operator for a system consisting of bosons and fermions.

We mostly use the language of statistics in this subsection.

**One-Dimensional Case** We start with the simplest case of a single real random variable $X$ which obeys a Gaussian distribution. Denote the expectation value of some function $f$ of the random variable $X$ with respect to this distribution by $\langle f(X)\rangle$. The expectation value of the $n$th power of the random variable itself, $\langle X^n\rangle$, is called *(raw) moment of order $n$*. A distinct feature of a Gaussian distribution is that it depends on two parameters only: On the *mean* $\mu = \langle X\rangle \in \mathbb{R}$, corresponding to the first moment of $X$, and on the *variance* $\sigma^2 = \langle X^2\rangle - \langle X\rangle^2 > 0$. Equivalently, one can say that only the two lowest moments are independent, i. e. $\langle X\rangle = \mu$ and $\langle X^2\rangle = \mu^2 + \sigma^2$. For instance,

[38]There are two cases in which the EoMs of the auxiliary fields are in fact free: For a noninteracting theory (trivially), and for the full (i. e. untruncated) interacting theory.

employing Wick's theorem (also known as *Isserlis' theorem* in statistics), one easily finds that $\langle X^3\rangle = \langle X\rangle^3 + 3\langle X\rangle\langle X^2\rangle = \mu^3 + 3\mu\sigma^2$ and $\langle X^4\rangle = \langle X\rangle^4 + 6\langle X\rangle^2\langle X^2\rangle + 3\langle X^2\rangle^2 = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$. In general, one has for powers of the *central* moment (corresponding to a *connected* correlator)

$$\langle (X-\mu)^n\rangle = \begin{cases} 0\,; & n \text{ is odd,} \\ (n-1)!!\,\sigma^n\,; & n \text{ is even,} \end{cases}$$

where $(2n-1)!! = (2n)!/(n!\,2^n)$ is the double factorial. It then follows easily by Taylor expanding that

$$\langle f(X)\rangle = \sum_{n=0}^{\infty} \frac{f^{(2n)}(\mu)}{n!\,2^n}\,\sigma^{2n} = f(\mu) + \frac{1}{2}\,f''(\mu)\,\sigma^2 + \frac{1}{8}\,f^{(4)}(\mu)\,\sigma^4 + \ldots = \tilde{f}(\mu,\sigma)\,,$$

i. e. the expectation value of $f$ fluctuates around the value of $f$ for the mean of $X$, and the higher terms in the series measure the failure of the commutation of applying $f$ and taking the expectation value. Note that $\langle f(X)\rangle$ depends on two quantities only: the mean $\mu$ and the *standard deviation* $\sigma$.

**Two-Dimensional Case** Similarly, a distribution of two random variables $X, Y$ is uniquely defined by the raw moments

$$\langle X\rangle\,, \quad \langle Y\rangle\,, \quad \langle X^2\rangle\,, \quad \langle Y^2\rangle\,, \quad \langle X\,Y\rangle\,,$$

or equivalently by the central moments

$$\begin{aligned} \mu_X &= \langle X\rangle\,, \\ \mu_Y &= \langle Y\rangle\,, \\ \sigma_X^2 &= \langle (X-\mu_x)^2\rangle = \langle X^2\rangle - \langle X\rangle^2\,, \\ \sigma_Y^2 &= \langle (Y-\mu_Y)^2\rangle = \langle Y^2\rangle - \langle Y\rangle^2\,, \\ \rho_{X,Y}\sigma_X\sigma_Y &= \langle (X-\mu_X)(Y-\mu_Y)\rangle = \langle X\,Y\rangle - \langle X\rangle\langle Y\rangle\,, \end{aligned}$$

i. e. by the means $\mu_X = \langle X\rangle$ and $\mu_Y = \langle Y\rangle$, the standard deviations $\sigma_X = \sqrt{\langle (X-\mu_X)^2\rangle}$ and $\sigma_Y = \sqrt{\langle (Y-\mu_Y)^2\rangle}$, and the *correlation* $\rho_{X,Y} = \langle (X-\mu_X)(Y-\mu_Y)\rangle/(\sigma_X\sigma_Y)$ of $X$ and $Y$.

By Taylor expanding some function $f$ of the two random variables around their means, so that we obtain an expansion in terms of the central moments, we then have

$$\begin{aligned} \langle f(X,Y)\rangle &= f(\mu_X,\mu_Y) + \frac{1}{2}\,\frac{\partial^2 f(X,Y)}{\partial X^2}\bigg|_{X=\mu_X,Y=\mu_Y} \sigma_X^2 + \frac{1}{2}\,\frac{\partial^2 f(X,Y)}{\partial Y^2}\bigg|_{X=\mu_X,Y=\mu_Y} \sigma_Y^2 + \ldots \\ &\quad + \frac{\partial^2 f(X,Y)}{\partial X \partial Y}\bigg|_{X=\mu_X,Y=\mu_Y} \rho_{X,Y}\sigma_X\sigma_Y + \ldots \\ &= \tilde{f}(\mu_X,\mu_Y,\sigma_X,\sigma_Y,\rho_{X,Y})\,. \end{aligned}$$

Note that each moment $\langle x^m y^n\rangle$, and hence in particular the higher terms in the above expansion, depends only on these five quantities.

### Infinite Dimensional Gaussian Distributions of Quantum Fields

Although quantum fields have an infinite[39] number of DoFs, they can be traded analogously to the one- and two-dimensional cases considered in the previous subsection (depending on whether they are fermions or bosons), since the field values at different positions in spacetime are uncorrelated.[40] In the following, we will always assume $x^0 = 0$.

**Bosonic Quantum Fields** Since the EoM of a bosonic quantum field $\Phi$ is a second-order partial differential equation, its unique solution is specified by providing $\Phi(x)|_{x^0=0}$ and $\partial_\mu \Phi(x)|_{x^0=0}$, which corresponds to five random variables. Symmetries, however, can reduce the number of independent random variables. For instance, in vacuum, after a Fourier transformation we have $\Phi(x) \to \Phi(p)$, $\partial_\mu \Phi(x) \to -\mathrm{i}\, p_\mu \Phi(p)$, so that there is only one independent random variable, namely $\Phi(p)$. We then have the analogy $X \to \Phi(p)$ to the case of the one-dimensional Gaussian distribution considered in the previous subsection.

For a spatially homogeneous and isotropic state, we have $\Phi(x) \to \Phi(x^0; \boldsymbol{p})$ and $\partial_\mu \Phi(x) \to (\delta^0_\mu \partial/\partial x^0 - \mathrm{i}\, \delta^i_\mu p_i)\Phi(x^0; \boldsymbol{p})$, so that there are two independent random variables, namely $\Phi(0; \boldsymbol{p})$ and $\partial\Phi(x^0; \boldsymbol{p})/\partial x^0|_{x^0=0}$. We then have the analogy

$$X \to \Phi(0; \boldsymbol{p}) \,, \quad Y \to \frac{\partial}{\partial x^0} \Phi(x^0; \boldsymbol{p}) \bigg|_{x^0=0}$$

to the case of the two-dimensional Gaussian distribution considered in the previous subsection. The distribution then depends on the quantities

$$\begin{gathered}
\langle \Phi(0; \boldsymbol{p}) \rangle \,, \quad \frac{\partial}{\partial x^0} \langle \Phi(x^0; \boldsymbol{p}) \rangle \bigg|_{x^0=0} \,, \\
\langle \Phi(0; \boldsymbol{p}) \Phi(0; -\boldsymbol{p}) \rangle - \langle \Phi(0; \boldsymbol{p}) \rangle \langle \Phi(0; -\boldsymbol{p}) \rangle \,, \\
\frac{\partial}{\partial x^0} \frac{\partial}{\partial y^0} \Big[ \langle \Phi(x^0; \boldsymbol{p}) \Phi(y^0; -\boldsymbol{p}) \rangle - \langle \Phi(x^0; \boldsymbol{p}) \rangle \langle \Phi(y^0; -\boldsymbol{p}) \rangle \Big] \bigg|_{x^0=y^0=0} \,, \\
\frac{1}{2} \left[ \frac{\partial}{\partial x^0} \langle \Phi(x^0; \boldsymbol{p}) \Phi(0; -\boldsymbol{p}) \rangle + \frac{\partial}{\partial y^0} \langle \Phi(0; \boldsymbol{p}) \Phi(y^0; -\boldsymbol{p}) \rangle \right] \bigg|_{x^0=y^0=0} \,,
\end{gathered}$$

where $\langle \cdot \rangle = \mathrm{Tr}(\rho \,\cdot)$ and in the last line we have used $\langle X\, Y \rangle \to \langle X\, Y + Y\, X \rangle/2$ since $X$ and $Y$ correspond to quantum fields here and hence do not commute. Note that

$$\begin{aligned}
\langle \Phi(x^0; \boldsymbol{p}) \Phi(y^0; -\boldsymbol{p}) \rangle = \;& \frac{1}{2} \langle \Phi(x^0; \boldsymbol{p}) \Phi(y^0; -\boldsymbol{p}) + \Phi(y^0; -\boldsymbol{p}) \Phi(x^0; \boldsymbol{p}) \rangle \\
&+ \frac{1}{2} \langle \Phi(x^0; \boldsymbol{p}) \Phi(y^0; -\boldsymbol{p}) - \Phi(y^0; -\boldsymbol{p}) \Phi(x^0; \boldsymbol{p}) \rangle \\
&= \frac{1}{2} \langle \{ \Phi(x^0; \boldsymbol{p}), \Phi(y^0; -\boldsymbol{p}) \} \rangle + \frac{1}{2} \langle [ \Phi(x^0; \boldsymbol{p}), \Phi(y^0; -\boldsymbol{p}) ] \rangle \\
&= F(x^0, y^0; \boldsymbol{p}) - \frac{\mathrm{i}}{2} \rho(x^0, y^0; \boldsymbol{p}) \,.
\end{aligned}$$

[39] in fact, even uncountable

[40] As are possible components of the quantum field, like for the photon field.

The equal-time spectral function $\rho$ and its first and second derivatives, however, are determined by the canonical commutation relations (see the next section), so that the only independent initial conditions are the equal-time statistical function and its first and second derivatives (in addition to the field expectation value and its first derivative). We therefore have the following independent random variables:

$$\phi(0;\boldsymbol{p})\,, \qquad \frac{\partial}{\partial x^0}\phi(x^0;\boldsymbol{p})\bigg|_{x^0=0}\,,$$
$$F(0,0;\boldsymbol{p})\,, \qquad \frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}F(x^0,y^0;\boldsymbol{p})\bigg|_{x^0=y^0=0}\,, \qquad \frac{1}{2}\left[\frac{\partial}{\partial x^0}F(x^0,0;\boldsymbol{p})+\frac{\partial}{\partial y^0}F(0,y^0;\boldsymbol{p})\right]\bigg|_{x^0=y^0=0}$$

with $\phi = \langle\Phi\rangle$. These five quantities uniquely determine the respective Gaussian density operator and hence the initial state of the system.

Note that for the case of vanishing field expectation value, the initial state of the system depends only on the last three terms involving the statistical function.

**Fermionic Quantum Fields** Since the EOM of a fermionic quantum field $\Psi$ is a first-order partial differential equation, its unique solution is specified by providing $\Psi(x)|_{x^0=0}$, which corresponds to a single random variable. We then have the analogy

$$X \to \Psi(0;\boldsymbol{p})$$

to the case of a one-dimensional random variable. The distribution then depends on the quantities

$$\langle\Psi(0;\boldsymbol{p})\rangle\,, \quad \langle\Psi(0;\boldsymbol{p})\Psi(0;-\boldsymbol{p})\rangle - \langle\Psi(0;\boldsymbol{p})\rangle\langle\Psi(0;-\boldsymbol{p})\rangle\,. \tag{3.77}$$

Since the two-point correlator can again be decomposed into statistical and spectral components, where the equal-time value of the spectral function is determined by the canonical anticommutation relations, we therefore have the following independent random variables:

$$\psi(0;\boldsymbol{p})\,, \qquad F^{(\mathrm{f})}(0,0;\boldsymbol{p}) \tag{3.78}$$

with $\psi = \langle\Psi\rangle$. These two quantities uniquely determine the respective Gaussian density operator and hence the initial state of the system.

Note that for the case of vanishing field expectation value, the initial state of the system depends only on the last term involving the statistical function.

**QED** Since bosonic and fermionic quantum fields are uncorrelated (as is the case for different components of bosonic fields like the photon field), a Gaussian density operator describing the initial state of the evolution of a QED system depends on the following quantities:

$$F^{(\mathrm{g})}_{\mu\nu}(0,0;\boldsymbol{p})\,, \quad \frac{\partial}{\partial x^0}F^{(\mathrm{g})}_{\mu\nu}(x^0,0;\boldsymbol{p})\bigg|_{x^0=0}\,, \quad \frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}F^{(\mathrm{g})}_{\mu\nu}(x^0,y^0;\boldsymbol{p})\bigg|_{x^0=y^0=0}\,,$$
$$F^{(\mathrm{f})}(0,0;\boldsymbol{p})\,,$$

where we have used the symmetry property of the photon statistical function under exchange of its time arguments and assumed vanishing field expectation values for physical reasons.

Although one is in principle completely free to choose the initial conditions for the statistical functions, it is convenient to assume that at initial time, the statistical functions have a thermal form, i. e. can be written as the sum of a free vacuum part and a free "thermal" part. The thermal part introduces a distribution function which one can then choose arbitrarily instead of a thermal one. The arbitrariness of the initial conditions of the statistical function is then shifted to the choice of an initial distribution function. This has the advantage of providing a simple interpretation of the system at initial time.

There are essentially two ways to derive the expression for the initial value of a statistical function which has the form of a thermal one and hence depends on some distribution function. One can either start with the spectral function in Fourier space and employ the fluctuation-dissipation relation [Ber05]. If we consider the photon case for definiteness, one then has

$$F^{(\mathrm{g})}_{\mu\nu}(0,0;\boldsymbol{p}) = -\,\mathrm{i}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_0}{2\pi}\,\left[\frac{1}{2}+n^{(\mathrm{g})}(\boldsymbol{p})\right]\rho^{(\mathrm{g})}_{\mu\nu}(p_0,\boldsymbol{p})\,, \tag{3.79}$$

which is usually rather easy to calculate since the free spectral function is proportional to a delta distribution. This is the method we will use in this work.

Alternatively, one can use that for equal times (and hence in particular at initial time), the statistical function is equal to the Feynman propagator, so that

$$F^{(\mathrm{g})}_{\mu\nu}(0,0;\boldsymbol{p}) = D_{\mu\nu}(0,0;\boldsymbol{p}) = \int_{-\infty}^{\infty}\frac{\mathrm{d}p_0}{2\pi}\,D_{\mu\nu}(p_0,\boldsymbol{p})\,. \tag{3.80}$$

By changing to Euclidean momentum or applying the residue theorem, this is also usually easy to calculate. We will, however, only present the first method in the following. Of course, it is easy (though tedious) to check that both methods yield the same results.

### 3.4.2 Photon Initial Conditions

#### Spectral Function

The initial condition (and the values at equal times in general) of the spectral function is fixed by the equal-time canonical commutation relations. The nonvanishing ones read:

- Without NL field:

  $$[A_\mu(x),\Pi^\nu(y)]|_{x^0=y^0} = \mathrm{i}\,\delta_\mu^\nu\delta^3(\boldsymbol{x}-\boldsymbol{y})\,,$$

  where

  $$\Pi^\mu(x) = \frac{\partial\mathcal{L}}{\partial\dot{A}_\mu(x)} = F^{\mu 0}(x) - \frac{1}{\xi}\,g^{\mu 0}\partial^\nu A_\nu(x)$$

  is the conjugate momentum of the photon field.
- With NL field:

  $$[A_i(x),\Pi^j(y)]|_{x^0=y^0} = \mathrm{i}\,\delta_i^j\delta^3(\boldsymbol{x}-\boldsymbol{y})\,,\quad [B(x),\Pi_B(y)]|_{x^0=y^0} = \mathrm{i}\,\delta^3(\boldsymbol{x}-\boldsymbol{y})\,,$$

where

$$\Pi^i(x) = \frac{\partial \mathcal{L}_{\mathrm{NL}}}{\partial \dot{A}_i(x)} = F^{i0}(x)\,, \quad \Pi_B(x) = \frac{\partial \mathcal{L}_{\mathrm{NL}}}{\partial \dot{B}(x)} = -A_0(x)$$

are the conjugate momenta of the spatial components of the photon field and of the NL field, respectively.

Of course, both methods are equivalent, and the connection is given by

$$(A_\mu(x)) = (-\Pi_B(x), \boldsymbol{A}(x))\,, \qquad (\Pi^\mu(x)) = \begin{pmatrix} B(x) \\ \boldsymbol{\Pi}(x) \end{pmatrix}\,.$$

The initial conditions for the photon spectral function can then be found by evaluating the equal-time commutation relations. For instance, one has

$$0 = [A_\mu(x), A_\nu(y)]|_{x^0=y^0} = \langle [A_\mu(x), A_\nu(y)] \rangle|_{x^0=y^0} = -\,\mathrm{i}\,\rho^{(\mathrm{g})}_{\mu\nu}(x,y)|_{x^0=y^0}\,,$$

so it immediately follows that $\rho^{(\mathrm{g})}_{\mu\nu}(x,y)|_{x^0=y^0} = 0$ or $\rho^{(\mathrm{g})}_{\mu\nu}(0,0;\boldsymbol{p}) = 0$.

Further, we have:

$$\begin{aligned}
\mathrm{i}\,\delta^3(\boldsymbol{x}-\boldsymbol{y}) = \langle [A_0(x), \Pi^0(y)] \rangle|_{x^0=y^0} &= \left\langle \left[ A_0(x), -\frac{1}{\xi}\,\partial^\mu A_\mu(y) \right] \right\rangle \Big|_{x^0=y^0} \\
&= -\frac{1}{\xi}\,\partial^\mu_y \langle [A_0(x), A_\mu(y)] \rangle|_{x^0=y^0} = \frac{\mathrm{i}}{\xi}\,\partial^\mu_y \rho^{(\mathrm{g})}_{0\mu}(x,y)|_{x^0=y^0}\,,
\end{aligned}$$

so $\partial^\mu_y \rho^{(\mathrm{g})}_{0\mu}(x,y)|_{x^0=y^0} = \xi\,\delta^3(\boldsymbol{x}-\boldsymbol{y})$, or

$$\left.\frac{\rho^{(\mathrm{g})}_{00}(0,y^0;\boldsymbol{p})}{\partial y^0}\right|_{y^0=0} + \mathrm{i}\,p^i \rho^{(\mathrm{g})}_{0i}(0,0;\boldsymbol{p}) = \xi\,.$$

One then finds:

$$\begin{aligned}
\rho^{(\mathrm{g})}_{\mu\nu}(x^0,y^0;\boldsymbol{p})|_{x^0=y^0} &= 0\,, \\
\left.\frac{\partial}{\partial x^0}\,\rho^{(\mathrm{g})}_{\mu\nu}(x^0,y^0;\boldsymbol{p})\right|_{x^0=y^0} &= -\Big[ g_{\mu\nu} - (1-\xi)\,\delta^0_\mu\,\delta^0_\nu \Big] = -\Big( \xi\,\delta^0_\mu\delta^0_\nu + \delta^i_\mu\delta^j_\nu\,g_{ij} \Big)\,, \\
\left.\frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}\,\rho^{(\mathrm{g})}_{\mu\nu}(x^0,y^0;\boldsymbol{p})\right|_{x^0=y^0} &= \mathrm{i}(1-\xi)\Big( \delta^0_\mu\delta^i_\nu + \delta^i_\mu\delta^0_\nu \Big)\,p_i\,,
\end{aligned} \tag{3.81}$$

or in terms of the isotropic components:[41]

$$
\begin{aligned}
\rho_{\mathrm{S}}^{(\mathrm{g})}(x^0,y^0;p)|_{x^0=y^0} &= 0\,, & \rho_{\mathrm{V}}^{(\mathrm{g})}(x^0,y^0;p)|_{x^0=y^0} &= 0\,,\\
\rho_{\mathrm{T}}^{(\mathrm{g})}(x^0,y^0;p)|_{x^0=y^0} &= 0\,, & \rho_{\mathrm{L}}^{(\mathrm{g})}(x^0,y^0;p)|_{x^0=y^0} &= 0\,,\\
\frac{\partial}{\partial x^0}\rho_{\mathrm{S}}^{(\mathrm{g})}(x^0,y^0;p)\bigg|_{x^0=y^0} &= -\xi\,, & \frac{\partial}{\partial x^0}\rho_{\mathrm{V}}^{(\mathrm{g})}(x^0,y^0;p)\bigg|_{x^0=y^0} &= 0\,,\\
\frac{\partial}{\partial x^0}\rho_{\mathrm{T}}^{(\mathrm{g})}(x^0,y^0;p)\bigg|_{x^0=y^0} &= -1\,, & \frac{\partial}{\partial x^0}\rho_{\mathrm{L}}^{(\mathrm{g})}(x^0,y^0;p)\bigg|_{x^0=y^0} &= -1\,,\\
\frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}\rho_{\mathrm{S}}^{(\mathrm{g})}(x^0,y^0;p)\bigg|_{x^0=y^0} &= 0\,, & \frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}\rho_{\mathrm{V}}^{(\mathrm{g})}(x^0,y^0;p)\bigg|_{x^0=y^0} &= -(1-\xi)\,p\,,\\
\frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}\rho_{\mathrm{T}}^{(\mathrm{g})}(x^0,y^0;p)\bigg|_{x^0=y^0} &= 0\,, & \frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}\rho_{\mathrm{L}}^{(\mathrm{g})}(x^0,y^0;p)\bigg|_{x^0=y^0} &= 0\,.
\end{aligned}
$$

If the auxiliary field is introduced, we also need the equal-time values for correlators involving the auxiliary field. One finds:

$$
\begin{aligned}
\rho_\mu^{(BA)}(x^0,y^0;\boldsymbol{p})|_{x^0=y^0} &= \delta_\mu^0\,,\\
\frac{\partial}{\partial x^0}\rho_\mu^{(BA)}(x^0,y^0;\boldsymbol{p})\bigg|_{x^0=y^0} &= -\,\mathrm{i}\,\delta_\mu^i p_i\,,\\
\frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}\rho_\mu^{(BA)}(x^0,y^0;\boldsymbol{p})\bigg|_{x^0=y^0} &= \delta_\mu^0\,\boldsymbol{p}^2
\end{aligned}
\tag{3.82}
$$

and

$$
\begin{aligned}
\rho_\mu^{(AB)}(x^0,y^0;\boldsymbol{p})|_{x^0=y^0} &= -\delta_\mu^0\,,\\
\frac{\partial}{\partial x^0}\rho_\mu^{(AB)}(x^0,y^0;\boldsymbol{p})\bigg|_{x^0=y^0} &= \mathrm{i}\,\delta_\mu^i p_i\,,\\
\frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}\rho_\mu^{(AB)}(x^0,y^0;\boldsymbol{p})\bigg|_{x^0=y^0} &= -\delta_\mu^0\,\boldsymbol{p}^2\,,
\end{aligned}
\tag{3.83}
$$

as well as

$$
\begin{aligned}
\rho^{(BB)}(x^0,y^0;\boldsymbol{p})|_{x^0=y^0} &= 0\,,\\
\frac{\partial}{\partial x^0}\rho^{(BB)}(x^0,y^0;\boldsymbol{p})\bigg|_{x^0=y^0} &= 0\,,\\
\frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}\rho^{(BB)}(x^0,y^0;\boldsymbol{p})\bigg|_{x^0=y^0} &= 0\,.
\end{aligned}
\tag{3.84}
$$

[41] Since at equal times the values of the vector components are identical, we define $\rho_{\mathrm{V}}^{(\mathrm{g})}(x^0,y^0;p)|_{x^0=y^0} := \widetilde{\rho}_{\mathrm{V}_1}^{(\mathrm{g})}(x^0,y^0;p)|_{x^0=y^0} = \widetilde{\rho}_{\mathrm{V}_2}^{(\mathrm{g})}(x^0,y^0;p)|_{x^0=y^0}$ and similarly for the derivatives.

The initial conditions then follow for $x^0 = y^0 = 0$.
Note that the equal-time values of the spectral function correspond to the sum rules

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, \rho^{(\mathrm{g})}_{\mu\nu}(p_0,\boldsymbol{p}) &= 0\,, \\ \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, p_0\, \rho^{(\mathrm{g})}_{\mu\nu}(p_0,\boldsymbol{p}) &= -\,\mathrm{i}\Big(\xi\, \delta^0_\mu \delta^0_\nu + \delta^i_\mu \delta^j_\nu\, g_{ij}\Big)\,, \\ \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, p_0^2\, \rho^{(\mathrm{g})}_{\mu\nu}(p_0,\boldsymbol{p}) &= \mathrm{i}(1-\xi)\Big(\delta^0_\mu \delta^i_\nu + \delta^i_\mu \delta^0_\nu\Big)\, p_i\,, \end{aligned} \tag{3.85}$$

or in terms of the isotropic components:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, \rho^{(\mathrm{g})}_{\mathrm{S}}(p_0,p) &= 0\,, & \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, \rho^{(\mathrm{g})}_{\mathrm{V}}(p_0,p) &= 0\,, \\ \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, \rho^{(\mathrm{g})}_{\mathrm{L}}(p_0,p) &= 0\,, & \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, \rho^{(\mathrm{g})}_{\mathrm{T}}(p_0,p) &= 0\,, \\ \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, p_0\, \rho^{(\mathrm{g})}_{\mathrm{S}}(p_0,p) &= -\,\mathrm{i}\,\xi\,, & \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, p_0\, \rho^{(\mathrm{g})}_{\mathrm{V}}(p_0,p) &= 0\,, \\ \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, p_0\, \rho^{(\mathrm{g})}_{\mathrm{L}}(p_0,p) &= -\,\mathrm{i}\,, & \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, p_0\, \rho^{(\mathrm{g})}_{\mathrm{T}}(p_0,p) &= -\,\mathrm{i}\,, \\ \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, p_0^2\, \rho^{(\mathrm{g})}_{\mathrm{S}}(p_0,p) &= 0\,, & \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, p_0^2\, \rho^{(\mathrm{g})}_{\mathrm{V}}(p_0,p) &= (1-\xi)\,p\,, \\ \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, p_0^2\, \rho^{(\mathrm{g})}_{\mathrm{L}}(p_0,p) &= 0\,, & \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, p_0^2\, \rho^{(\mathrm{g})}_{\mathrm{T}}(p_0,p) &= 0\,. \end{aligned}$$

**Statistical Function**

With

$$D^{>}_{\mu\nu}(x,y) = \langle A_\mu(x) A_\nu(y)\rangle\,, \quad D^{<}_{\mu\nu}(x,y) = -\langle A_\nu(y) A_\mu(x)\rangle\,,$$

one has in thermal equilibrium [LB00]

$$\mathrm{i}\, D^{>}_{\mu\nu}(p) = \Big[1 + n_{\mathrm{BE}}(p_0)\Big]\rho^{(\mathrm{g})}_{\mu\nu}(p)\,, \quad \mathrm{i}\, D^{<}_{\mu\nu}(p) = n_{\mathrm{BE}}(p_0)\rho^{(\mathrm{g})}_{\mu\nu}(p)\,,$$

where

$$n_{\mathrm{BE}}(p_0) = \frac{1}{\mathrm{e}^{\beta p_0} - 1} \tag{3.86}$$

is the Bose–Einstein distribution function at inverse temperature $\beta$. Then:

$$F^{(\mathrm{g})}_{\mu\nu}(p) = \frac{1}{2}\Big[D^{>}_{\mu\nu}(p) + D^{<}_{\mu\nu}(p)\Big] = -\,\mathrm{i}\left[\frac{1}{2} + n_{\mathrm{BE}}(p_0)\right]\rho^{(\mathrm{g})}_{\mu\nu}(p)\,.$$

This is the fluctuation-dissipation relation for photons, and what is remarkable about it is that the distribution function *in thermal equilibrium* depends only on $p_0$, but not on $\boldsymbol{p}$. Out-of-equilibrium, a similar relation can be written down, by replacing the thermal distribution function with some function $\widetilde{n}^{(\mathrm{g})}$ which in general depends on $\boldsymbol{p}$ as well. Note that this does not involve any assumption. We then have:

$$F^{(\mathrm{g})}_{\mu\nu}(p) = -\,\mathrm{i}\left[\frac{1}{2} + \widetilde{n}^{(\mathrm{g})}(p)\right]\rho^{(\mathrm{g})}_{\mu\nu}(p)\,.$$

We can now derive a relation involving $\widetilde{n}^{(\mathrm{g})}$ by making use of the symmetry properties of the statistical and spectral functions. Since the statistical function is even and the spectral function is odd under inversion of the frequency, it follows that

$$F^{(\mathrm{g})}_{\mu\nu}(-p_0,\boldsymbol{p}) = \mathrm{i}\left[\frac{1}{2} + \widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p})\right]\rho^{(\mathrm{g})}_{\mu\nu}(p_0,\boldsymbol{p})\,.$$

On the other hand, by just inverting $p_0$ in each quantity, we obtain

$$F^{(\mathrm{g})}_{\mu\nu}(-p_0,\boldsymbol{p}) = -\,\mathrm{i}\left[\frac{1}{2} + \widetilde{n}^{(\mathrm{g})}(-p_0,\boldsymbol{p})\right]\rho^{(\mathrm{g})}_{\mu\nu}(-p_0,\boldsymbol{p})\,.$$

By comparison, it immediately follows that

$$\frac{1}{2} + \widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p}) = -\left[\frac{1}{2} + \widetilde{n}^{(\mathrm{g})}(-p_0,\boldsymbol{p})\right] \quad \text{or} \quad 1 + \widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p}) + \widetilde{n}^{(\mathrm{g})}(-p_0,\boldsymbol{p}) = 0\,.$$

It can easily be checked that any solution to this equation can be parametrized as

$$\widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p}) = \frac{1}{\mathrm{e}^{\beta(\boldsymbol{p})p_0} - 1}\,.$$

Note the close resemblance to the Bose–Einstein distribution (3.86). The only, albeit crucial, difference is that instead of on a single number $\beta$ (the inverse temperature), we now have a dependence on an (essentially[42]) arbitrary function $\beta(\boldsymbol{p})$ which depends on the spatial momentum and could be called "mode temperature" [Ber05].[43]

Defining

$$\widetilde{n}^{(\mathrm{g})\prime}(p_0,\boldsymbol{p}) = \frac{\partial}{\partial p_0}\widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p})\,,$$

it is another easy exercise to show that

$$\begin{aligned}\frac{1}{2}\left[\widetilde{n}^{(\mathrm{g})\prime}(p_0,\boldsymbol{p}) + \widetilde{n}^{(\mathrm{g})\prime}(-p_0,\boldsymbol{p})\right] &= \frac{1}{p_0}\ln\left(1 + \frac{1}{\widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p})}\right)\widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p})\,\widetilde{n}^{(\mathrm{g})}(-p_0,\boldsymbol{p}) \\ &= -\frac{1}{p_0}\ln\left(1 + \frac{1}{\widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p})}\right)\widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p})\left[1 + \widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p})\right],\end{aligned}$$

and

$$\widetilde{n}^{(\mathrm{g})\prime}(p_0,\boldsymbol{p}) - \widetilde{n}^{(\mathrm{g})\prime}(-p_0,\boldsymbol{p}) = 0\,,$$

[42] We require $\widetilde{n}^{(\mathrm{g})}$ to be a distribution function on-shell, i. e. $p_0 = |\boldsymbol{p}|$. Since distribution functions must be nonnegative, we have $\widetilde{n}^{(\mathrm{g})}(|\boldsymbol{p}|,\boldsymbol{p}) \geq 0$ or $\beta(\boldsymbol{p}) \geq 0$, i. e. $\beta(\boldsymbol{p})$ must be nonnegative as well.

[43] Although it might be misleading to talk of a temperature out-of-equilibrium. It does, however, share the property of being nonnegative with the (thermal) temperature.

which we will need below.

Then:

$$
\begin{aligned}
&F^{(\mathrm{g})}_{\mu\nu}(0,0;\boldsymbol{p}) \\
&\quad = -\mathrm{i}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_0}{2\pi}\left[\frac{1}{2}+\widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p})\right]\rho^{(\mathrm{g})}_{\mu\nu}(p_0,\boldsymbol{p}) \\
&\quad = -\frac{\mathrm{i}}{2}\underbrace{\rho^{(\mathrm{g})}_{\mu\nu}(0,0;\boldsymbol{p})}_{=0\,,\ \text{see } (3.81)}-\mathrm{i}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_0}{2\pi}\,\widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p})\rho^{(\mathrm{g})}_{\mu\nu}(p_0,\boldsymbol{p}) \\
&\quad = -\mathrm{i}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_0}{2\pi}\,\widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p})\rho^{(\mathrm{g})}_{\mu\nu}(p_0,\boldsymbol{p}) \\
&\quad = -\int_{-\infty}^{\infty}\mathrm{d}p_0\,\mathrm{sgn}(p_0)\Big[g_{\mu\nu}\delta(p^2)+(1-\xi)p_\mu p_\nu\delta'(p^2)\Big]\,\widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p}) \\
&\quad = -\int_{-\infty}^{\infty}\mathrm{d}p_0\,\delta(p^2)\left\{g_{\mu\nu}+\frac{1-\xi}{2\boldsymbol{p}^2}\left[1-\mathrm{sgn}(p_0)|\boldsymbol{p}|\frac{\partial}{\partial p_0}\right]p_\mu p_\nu\right\}\mathrm{sgn}(p_0)\,\widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p}) \\
&\quad = -\int_{-\infty}^{\infty}\mathrm{d}p_0\,\delta(p^2)\left\{g_{\mu\nu}+\frac{1-\xi}{2\boldsymbol{p}^2}\left[p_\mu p_\nu-\mathrm{sgn}(p_0)|\boldsymbol{p}|\left(\delta^0_\mu p_\nu+\delta^0_\nu p_\mu+p_\mu p_\nu\frac{\partial}{\partial p_0}\right)\right]\right\} \\
&\qquad\qquad\qquad\qquad\qquad \cdot\,\mathrm{sgn}(p_0)\,\widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p}) \\
&\quad = -\int_{-\infty}^{\infty}\mathrm{d}p_0\,\delta(p^2)\left\{g_{\mu\nu}+\frac{1-\xi}{2\boldsymbol{p}^2}\left[p_\mu p_\nu\left(1-\mathrm{sgn}(p_0)|\boldsymbol{p}|\frac{\partial}{\partial p_0}\right)\right.\right. \\
&\qquad\qquad\qquad\qquad\left.\left.-\,\mathrm{sgn}(p_0)|\boldsymbol{p}|\Big(\delta^0_\mu p_\nu+\delta^0_\nu p_\mu\Big)\right]\right\}\mathrm{sgn}(p_0)\,\widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p}) \\
&\quad = -\int_{-\infty}^{\infty}\mathrm{d}p_0\,\delta(p^2)\left\{\mathrm{sgn}(p_0)g_{\mu\nu}\right. \\
&\qquad\qquad\left.+\,\frac{1-\xi}{2}\left[\frac{p_\mu p_\nu}{\boldsymbol{p}^2}\left(\mathrm{sgn}(p_0)-|\boldsymbol{p}|\frac{\partial}{\partial p_0}\right)-\frac{\delta^0_\mu p_\nu+\delta^0_\nu p_\mu}{|\boldsymbol{p}|}\right]\right\}\widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p}) \\
&\quad = f^{(1)}_{\mu\nu}(\boldsymbol{p})+f^{(2)}_{\mu\nu}(\boldsymbol{p})+f^{(3)}_{\mu\nu}(\boldsymbol{p})+f^{(4)}_{\mu\nu}(\boldsymbol{p})\,,
\end{aligned}
$$

where in the last but one line we have used that

$$
\delta(p^2)\frac{\partial}{\partial p_0}\,\mathrm{sgn}(p_0)=2\delta(p^2)\delta(p_0)=\frac{1}{|\boldsymbol{p}|}\Big[\delta(p_0-|\boldsymbol{p}|)+\delta(p_0+|\boldsymbol{p}|)\Big]\delta(p_0)=0\,.
$$

We will now calculate each of the terms $f^{(i)}_{\mu\nu}(\boldsymbol{p})$, $i=1,\ldots,4$, separately. The following identities will be useful:

$$
\begin{aligned}
p_\mu p_\nu|_{p_0=\pm|\boldsymbol{p}|} &= \Big(\pm\delta^0_\mu|\boldsymbol{p}|+\delta^i_\mu p_i\Big)\Big(\pm\delta^0_\nu|\boldsymbol{p}|+\delta^j_\nu p_j\Big) \\
&= \delta^0_\mu\delta^0_\nu\boldsymbol{p}^2\pm\delta^0_\mu\delta^i_\nu|\boldsymbol{p}|p_i\pm\delta^0_\nu\delta^i_\mu|\boldsymbol{p}|p_i+\delta^i_\mu\delta^j_\nu p_i p_j \\
&= \boldsymbol{p}^2\left[\delta^0_\mu\delta^0_\nu\pm\Big(\delta^0_\mu\delta^i_\nu+\delta^0_\nu\delta^i_\mu\Big)\frac{p_i}{|\boldsymbol{p}|}+\delta^i_\mu\delta^j_\nu\frac{p_i p_j}{\boldsymbol{p}^2}\right], \\
\Big(\delta^0_\mu p_\nu+\delta^0_\nu p_\mu\Big)\Big|_{p_0=\pm|\boldsymbol{p}|} &= \delta^0_\mu\Big(\pm\delta^0_\nu|\boldsymbol{p}|+\delta^i_\nu p_i\Big)+\delta^0_\nu\Big(\pm\delta^0_\mu|\boldsymbol{p}|+\delta^i_\mu p_i\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \pm 2\delta^0_\mu \delta^0_\nu |\boldsymbol{p}| + \delta^0_\mu \delta^i_\nu p_i + \delta^0_\nu \delta^i_\mu p_i \\
&= |\boldsymbol{p}| \left[ \pm 2\delta^0_\mu \delta^0_\nu + \left( \delta^0_\mu \delta^i_\nu + \delta^0_\nu \delta^i_\mu \right) \frac{p_i}{|\boldsymbol{p}|} \right] .
\end{aligned}
$$

Defining

$$
n^{(\mathrm{g})}(\boldsymbol{p}) = \widetilde{n}^{(\mathrm{g})}(|\boldsymbol{p}|, \boldsymbol{p}) ,
$$

which can be viewed as a nonequilibrium distribution function, we then have:

$$
\begin{aligned}
f^{(1)}_{\mu\nu}(\boldsymbol{p}) &= -g_{\mu\nu} \int_{-\infty}^{\infty} \mathrm{d}p_0 \, \mathrm{sgn}(p_0) \delta(p^2) \widetilde{n}^{(\mathrm{g})}(p_0, \boldsymbol{p}) \\
&= -\frac{g_{\mu\nu}}{2|\boldsymbol{p}|} \int_{-\infty}^{\infty} \mathrm{d}p_0 \Big[ \delta(p_0 - |\boldsymbol{p}|) - \delta(p_0 + |\boldsymbol{p}|) \Big] \widetilde{n}^{(\mathrm{g})}(p_0, \boldsymbol{p}) \\
&= -\frac{g_{\mu\nu}}{2|\boldsymbol{p}|} \Big[ n^{(\mathrm{g})}(|\boldsymbol{p}|, \boldsymbol{p}) - \widetilde{n}^{(\mathrm{g})}(-|\boldsymbol{p}|, \boldsymbol{p}) \Big] \\
&= -\frac{g_{\mu\nu}}{|\boldsymbol{p}|} \left[ \frac{1}{2} + \widetilde{n}^{(\mathrm{g})}(|\boldsymbol{p}|, \boldsymbol{p}) \right] \\
&= -\frac{1}{|\boldsymbol{p}|} \left[ \delta^0_\mu \delta^0_\nu + \delta^i_\mu \delta^j_\nu \left( g_{ij} + \frac{p_i p_j}{\boldsymbol{p}^2} \right) - \delta^i_\mu \delta^j_\nu \frac{p_i p_j}{\boldsymbol{p}^2} \right] \left[ \frac{1}{2} + n^{(\mathrm{g})}(\boldsymbol{p}) \right] , \\
f^{(2)}_{\mu\nu}(\boldsymbol{p}) &= -\frac{1-\xi}{2} \int_{-\infty}^{\infty} \mathrm{d}p_0 \, \mathrm{sgn}(p_0) \, \delta(p^2) \, \frac{p_\mu p_\nu}{\boldsymbol{p}^2} \widetilde{n}^{(\mathrm{g})}(p_0, \boldsymbol{p}) \\
&= -\frac{1-\xi}{2} \frac{1}{2|\boldsymbol{p}|} \int_{-\infty}^{\infty} \mathrm{d}p_0 \Big[ \delta(p_0 - |\boldsymbol{p}|) - \delta(p_0 + |\boldsymbol{p}|) \Big] \frac{p_\mu p_\nu}{\boldsymbol{p}^2} \widetilde{n}^{(\mathrm{g})}(p_0, \boldsymbol{p}) \\
&= -\frac{1-\xi}{2} \frac{1}{2|\boldsymbol{p}|} \Bigg\{ \quad \left[ \delta^0_\mu \delta^0_\nu + \left( \delta^0_\mu \delta^i_\nu + \delta^0_\nu \delta^i_\mu \right) \frac{p_i}{|\boldsymbol{p}|} + \delta^i_\mu \delta^j_\nu \frac{p_i p_j}{\boldsymbol{p}^2} \right] \widetilde{n}^{(\mathrm{g})}(|\boldsymbol{p}|, \boldsymbol{p}) \\
&\qquad - \left[ \delta^0_\mu \delta^0_\nu - \left( \delta^0_\mu \delta^i_\nu + \delta^0_\nu \delta^i_\mu \right) \frac{p_i}{|\boldsymbol{p}|} + \delta^i_\mu \delta^j_\nu \frac{p_i p_j}{\boldsymbol{p}^2} \right] \widetilde{n}^{(\mathrm{g})}(-|\boldsymbol{p}|, \boldsymbol{p}) \Bigg\} \\
&= -\frac{1-\xi}{2} \frac{1}{2|\boldsymbol{p}|} \Bigg\{ \left( \delta^0_\mu \delta^0_\nu + \delta^i_\mu \delta^j_\nu \frac{p_i p_j}{\boldsymbol{p}^2} \right) \Big[ \widetilde{n}^{(\mathrm{g})}(|\boldsymbol{p}|, \boldsymbol{p}) - \widetilde{n}^{(\mathrm{g})}(-|\boldsymbol{p}|, \boldsymbol{p}) \Big] \\
&\qquad + \left( \delta^0_\mu \delta^i_\nu + \delta^0_\nu \delta^i_\mu \right) \frac{p_i}{|\boldsymbol{p}|} \Big[ \underbrace{\widetilde{n}^{(\mathrm{g})}(|\boldsymbol{p}|, \boldsymbol{p}) + \widetilde{n}^{(\mathrm{g})}(-|\boldsymbol{p}|, \boldsymbol{p})}_{=-1} \Big] \Bigg\} \\
&= -\frac{1-\xi}{2} \frac{1}{|\boldsymbol{p}|} \left\{ \left( \delta^0_\mu \delta^0_\nu + \delta^i_\mu \delta^j_\nu \frac{p_i p_j}{\boldsymbol{p}^2} \right) \left[ \frac{1}{2} + \widetilde{n}^{(\mathrm{g})}(\boldsymbol{p}) \right] - \frac{1}{2} \left( \delta^0_\mu \delta^i_\nu + \delta^0_\nu \delta^i_\mu \right) \frac{p_i}{|\boldsymbol{p}|} \right\} , \\
f^{(3)}_{\mu\nu}(\boldsymbol{p}) &= \frac{1-\xi}{2} \int_{-\infty}^{\infty} \mathrm{d}p_0 \, \delta(p^2) \, \frac{p_\mu p_\nu}{|\boldsymbol{p}|} \frac{\partial}{\partial p_0} \widetilde{n}^{(\mathrm{g})}(p_0, \boldsymbol{p}) \\
&= \frac{1-\xi}{4} \int_{-\infty}^{\infty} \mathrm{d}p_0 \Big[ \delta(p_0 - |\boldsymbol{p}|) + \delta(p_0 + |\boldsymbol{p}|) \Big] \frac{p_\mu p_\nu}{\boldsymbol{p}^2} \frac{\partial}{\partial p_0} \widetilde{n}^{(\mathrm{g})}(p_0, \boldsymbol{p}) \\
&= \frac{1-\xi}{4} \Bigg\{ \quad \left[ \delta^0_\mu \delta^0_\nu + \left( \delta^0_\mu \delta^i_\nu + \delta^0_\nu \delta^i_\mu \right) \frac{p_i}{|\boldsymbol{p}|} + \delta^i_\mu \delta^j_\nu \frac{p_i p_j}{\boldsymbol{p}^2} \right] \widetilde{n}^{(\mathrm{g})\prime}(|\boldsymbol{p}|, \boldsymbol{p}) \\
&\qquad + \left[ \delta^0_\mu \delta^0_\nu - \left( \delta^0_\mu \delta^i_\nu + \delta^0_\nu \delta^i_\mu \right) \frac{p_i}{|\boldsymbol{p}|} + \delta^i_\mu \delta^j_\nu \frac{p_i p_j}{\boldsymbol{p}^2} \right] \widetilde{n}^{(\mathrm{g})\prime}(-|\boldsymbol{p}|, \boldsymbol{p}) \Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1-\xi}{4}\Bigg\{\left(\delta^0_\mu\delta^0_\nu + \delta^i_\mu\delta^j_\nu\frac{p_ip_j}{\boldsymbol{p}^2}\right)\left[\widetilde{n}^{(\mathrm{g})\prime}(|\boldsymbol{p}|,\boldsymbol{p}) + \widetilde{n}^{(\mathrm{g})\prime}(-|\boldsymbol{p}|,\boldsymbol{p})\right] \\
&\qquad + \left(\delta^0_\mu\delta^i_\nu + \delta^0_\nu\delta^i_\mu\right)\frac{p_i}{|\boldsymbol{p}|}\Big[\underbrace{\widetilde{n}^{(\mathrm{g})\prime}(|\boldsymbol{p}|,\boldsymbol{p}) - \widetilde{n}^{(\mathrm{g})\prime}(-|\boldsymbol{p}|,\boldsymbol{p})}_{=0}\Big]\Bigg\} \\
&= -\frac{1-\xi}{2}\left(\delta^0_\mu\delta^0_\nu + \delta^i_\mu\delta^j_\nu\frac{p_ip_j}{\boldsymbol{p}^2}\right)\ln\left(1+\frac{1}{n^{(\mathrm{g})}(\boldsymbol{p})}\right)n^{(\mathrm{g})}(\boldsymbol{p})\left[1+n^{(\mathrm{g})}(\boldsymbol{p})\right], \\
f^{(4)}_{\mu\nu}(\boldsymbol{p}) &= \frac{1-\xi}{2}\int_{-\infty}^{\infty}\mathrm{d}p_0\,\delta(p^2)\,\frac{\delta^0_\mu p_\nu + \delta^0_\nu p_\mu}{|\boldsymbol{p}|}\,\widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p}) \\
&= \frac{1-\xi}{2}\frac{1}{2|\boldsymbol{p}|}\int_{-\infty}^{\infty}\mathrm{d}p_0\left[\delta(p_0-|\boldsymbol{p}|)+\delta(p_0+|\boldsymbol{p}|)\right]\frac{\delta^0_\mu p_\nu + \delta^0_\nu p_\mu}{|\boldsymbol{p}|}\,\widetilde{n}^{(\mathrm{g})}(p_0,\boldsymbol{p}) \\
&= \frac{1-\xi}{2}\frac{1}{2|\boldsymbol{p}|}\Bigg\{\left[\;2\delta^0_\mu\delta^0_\nu + \left(\delta^0_\mu\delta^i_\nu+\delta^0_\nu\delta^i_\mu\right)\frac{p_i}{|\boldsymbol{p}|}\right]\widetilde{n}^{(\mathrm{g})}(|\boldsymbol{p}|,\boldsymbol{p}) \\
&\qquad + \left[-2\delta^0_\mu\delta^0_\nu + \left(\delta^0_\mu\delta^i_\nu+\delta^0_\nu\delta^i_\mu\right)\frac{p_i}{|\boldsymbol{p}|}\right]\widetilde{n}^{(\mathrm{g})}(-|\boldsymbol{p}|,\boldsymbol{p})\Bigg\} \\
&= \frac{1-\xi}{2}\frac{1}{2|\boldsymbol{p}|}\Bigg\{2\delta^0_\mu\delta^0_\nu\left[\widetilde{n}^{(\mathrm{g})}(|\boldsymbol{p}|,\boldsymbol{p}) - \widetilde{n}^{(\mathrm{g})}(-|\boldsymbol{p}|,\boldsymbol{p})\right] \\
&\qquad + \left(\delta^0_\mu\delta^i_\nu+\delta^0_\nu\delta^i_\mu\right)\frac{p_i}{|\boldsymbol{p}|}\left[\widetilde{n}^{(\mathrm{g})}(|\boldsymbol{p}|,\boldsymbol{p}) + \widetilde{n}^{(\mathrm{g})}(-|\boldsymbol{p}|,\boldsymbol{p})\right]\Bigg\} \\
&= \frac{1-\xi}{2}\frac{1}{|\boldsymbol{p}|}\left\{2\delta^0_\mu\delta^0_\nu\left[\frac{1}{2}+n^{(\mathrm{g})}(\boldsymbol{p})\right] - \frac{1}{2}\left(\delta^0_\mu\delta^i_\nu+\delta^0_\nu\delta^i_\mu\right)\frac{p_i}{|\boldsymbol{p}|}\right\}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&F^{(\mathrm{g})}_{\mu\nu}(0,0;\boldsymbol{p}) \\
&= \frac{1}{|\boldsymbol{p}|}\Bigg(\left\{-\frac{1+\xi}{2}\left[\frac{1}{2}+n^{(\mathrm{g})}(\boldsymbol{p})\right] - \frac{1-\xi}{2}\ln\left(1+\frac{1}{n^{(\mathrm{g})}(\boldsymbol{p})}\right)n^{(\mathrm{g})}(\boldsymbol{p})\left[1+n^{(\mathrm{g})}(\boldsymbol{p})\right]\right\}\delta^0_\mu\delta^0_\nu \\
&\qquad + \left\{\frac{1+\xi}{2}\left[\frac{1}{2}+n^{(\mathrm{g})}(\boldsymbol{p})\right] - \frac{1-\xi}{2}\ln\left(1+\frac{1}{n^{(\mathrm{g})}(\boldsymbol{p})}\right)n^{(\mathrm{g})}(\boldsymbol{p})\left[1+n^{(\mathrm{g})}(\boldsymbol{p})\right]\right\}\delta^i_\mu\delta^j_\nu\frac{p_ip_j}{\boldsymbol{p}^2} \\
&\qquad - \left[\frac{1}{2}+n^{(\mathrm{g})}(\boldsymbol{p})\right]\delta^i_\mu\delta^j_\nu\left(g_{ij}+\frac{p_ip_j}{\boldsymbol{p}^2}\right)\Bigg).
\end{aligned}
$$

The other initial conditions are obtained by similar calculations as

$$
\begin{aligned}
&\frac{\partial}{\partial x^0}F^{(\mathrm{g})}_{\mu\nu}(x^0,0;\boldsymbol{p})\bigg|_{x^0=0} \\
&\quad = -\mathrm{i}\int_{-\infty}^{\infty}\frac{\mathrm{d}p_0}{2\pi}\,p_0F^{(\mathrm{g})}_{\mu\nu}(p_0,\boldsymbol{p}) \\
&\quad = -\mathrm{i}\,\frac{1-\xi}{2}\left\{\frac{1}{2}+n^{(\mathrm{g})}(\boldsymbol{p}) + \ln\left(1+\frac{1}{n^{(\mathrm{g})}(\boldsymbol{p})}\right)n^{(\mathrm{g})}(\boldsymbol{p})\left[1+n^{(\mathrm{g})}(\boldsymbol{p})\right]\right\}(\delta^0_\mu\delta^i_\nu+\delta^0_\nu\delta^i_\mu)\frac{p_i}{|\boldsymbol{p}|}
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial}{\partial x^0}\,\frac{\partial}{\partial y^0} F^{(\mathrm{g})}_{\mu\nu}(x^0,y^0;\boldsymbol{p})\bigg|_{x^0=y^0=0} \\
&= \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, p_0^2 F^{(\mathrm{g})}_{\mu\nu}(p_0,\boldsymbol{p}) \\
&= |\boldsymbol{p}| \Bigg( \left\{ \frac{1-3\xi}{2}\left[\frac{1}{2}+n^{(\mathrm{g})}(\boldsymbol{p})\right] + \frac{1-\xi}{2}\ln\!\left(1+\frac{1}{n^{(\mathrm{g})}(\boldsymbol{p})}\right) n^{(\mathrm{g})}(\boldsymbol{p})\Big[1+n^{(\mathrm{g})}(\boldsymbol{p})\Big]\right\} \delta^0_\mu \delta^0_\nu \\
&\qquad + \left\{ \frac{3-\xi}{2}\left[\frac{1}{2}+n^{(\mathrm{g})}(\boldsymbol{p})\right] - \frac{1-\xi}{2}\ln\!\left(1+\frac{1}{n^{(\mathrm{g})}(\boldsymbol{p})}\right) n^{(\mathrm{g})}(\boldsymbol{p})\Big[1+n^{(\mathrm{g})}(\boldsymbol{p})\Big]\right\} \delta^i_\mu \delta^j_\nu \frac{p_i p_j}{\boldsymbol{p}^2} \\
&\qquad - \left[\frac{1}{2}+n^{(\mathrm{g})}(\boldsymbol{p})\right] \delta^i_\mu \delta^j_\nu \left(g_{ij}+\frac{p_i p_j}{\boldsymbol{p}^2}\right)\Bigg) .
\end{aligned}
$$

In terms of the Lorentz components, the initial conditions of the photon statistical function then read:

$$
\begin{aligned}
F^{(\mathrm{g})}_{\mathrm{S}}(0,0;p) &= -\frac{1}{p}\left\{ \frac{1+\xi}{2}\left[\frac{1}{2}+n^{(\mathrm{g})}(p)\right] - \frac{1-\xi}{2}\ln\!\left(1+\frac{1}{n^{(\mathrm{g})}(p)}\right) n^{(\mathrm{g})}(p)\left[1+n^{(\mathrm{g})}(p)\right]\right\} ,\\
F^{(\mathrm{g})}_{\mathrm{V}}(0,0;p) &= 0 ,\\
F^{(\mathrm{g})}_{\mathrm{T}}(0,0;p) &= -\frac{1}{p}\left[\frac{1}{2}+n^{(\mathrm{g})}(p)\right] ,\\
F^{(\mathrm{g})}_{\mathrm{L}}(0,0;p) &= -\frac{1}{p}\left\{ \frac{1+\xi}{2}\left[\frac{1}{2}+n^{(\mathrm{g})}(p)\right] + \frac{1-\xi}{2}\ln\!\left(1+\frac{1}{n^{(\mathrm{g})}(p)}\right) n^{(\mathrm{g})}(p)\left[1+n^{(\mathrm{g})}(p)\right]\right\} ,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x^0}\, F^{(\mathrm{g})}_{\mathrm{S}}(x^0,0;p)\bigg|_{x^0=0} &= 0 ,\\
\frac{\partial}{\partial x^0}\, F^{(\mathrm{g})}_{\mathrm{V}}(x^0,0;p)\bigg|_{x^0=0} &= -\,\mathrm{i}\,\frac{1-\xi}{2}\left\{\frac{1}{2}+n^{(\mathrm{g})}(p)+\ln\!\left(1+\frac{1}{n^{(\mathrm{g})}(p)}\right) n^{(\mathrm{g})}(p)\left[1+n^{(\mathrm{g})}(p)\right]\right\} ,\\
\frac{\partial}{\partial x^0}\, F^{(\mathrm{g})}_{\mathrm{T}}(x^0,0;p)\bigg|_{x^0=0} &= 0 ,\\
\frac{\partial}{\partial x^0}\, F^{(\mathrm{g})}_{\mathrm{L}}(x^0,0;p)\bigg|_{x^0=0} &= 0 ,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x^0}\,\frac{\partial}{\partial y^0}\, F^{(\mathrm{g})}_{\mathrm{S}}(x^0,y^0;p)\bigg|_{x^0=y^0=0} = p\,\Bigg\{ &\frac{1-3\xi}{2}\left[\frac{1}{2}+n^{(\mathrm{g})}(p)\right] \\
&+ \frac{1-\xi}{2}\ln\!\left(1+\frac{1}{n^{(\mathrm{g})}(p)}\right) n^{(\mathrm{g})}(p)\left[1+n^{(\mathrm{g})}(p)\right]\Bigg\} ,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}F_{\mathrm{V}}^{(\mathrm{g})}(x^0,y^0;p)\bigg|_{x^0=y^0=0} &= 0\,,\\
\frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}F_{\mathrm{T}}^{(\mathrm{g})}(x^0,y^0;p)\bigg|_{x^0=y^0=0} &= -p\left[\frac{1}{2}+n^{(\mathrm{g})}(p)\right]\,,\\
\frac{\partial}{\partial x^0}\frac{\partial}{\partial y^0}F_{\mathrm{L}}^{(\mathrm{g})}(x^0,y^0;p)\bigg|_{x^0=y^0=0} &= -p\left\{\frac{3-\xi}{2}\left[\frac{1}{2}+n^{(\mathrm{g})}(p)\right]\right.\\
&\qquad\left.+\frac{1-\xi}{2}\ln\!\left(1+\frac{1}{n^{(\mathrm{g})}(p)}\right)n^{(\mathrm{g})}(p)\left[1+n^{(\mathrm{g})}(p)\right]\right\}.
\end{aligned}
$$

### 3.4.3 Fermion Initial Conditions

Since the fermion EOMs are first-oder differential equations, their solutions are determined by a single initial condition, i. e. the value of the respective quantity at initial time.

#### Spectral Function

The initial condition for the fermion spectral function is determined by the canonical commutation relation

$$
\left\{\Psi(x),\Psi^\dagger(y)\right\}\Big|_{x^0=y^0} = \delta^3(\boldsymbol{x}-\boldsymbol{y})\,.
$$

From the definition of the fermion spectral function in terms of the fermion quantum field operators it follows that

$$
\rho^{(\mathrm{f})}(x,y)|_{x^0=y^0} = \mathrm{i}\,\gamma^0\delta^3(\boldsymbol{x}-\boldsymbol{y})\,,
$$

and by applying a partial Fourier transformation with respect to space,

$$
\rho^{(\mathrm{f})}(x^0,y^0;\boldsymbol{p})\Big|_{x^0=y^0} = \mathrm{i}\,\gamma^0\,. \tag{3.87}
$$

In terms of the Lorentz components, one has

$$
\begin{aligned}
\rho_{\mathrm{S}}^{(\mathrm{f})}(x^0,y^0;p)\Big|_{x^0=y^0} &= 0\,, \quad & \widetilde{\rho}_{\mathrm{V}}^{(\mathrm{f})0}(x^0,y^0;p)\Big|_{x^0=y^0} &= 1\,,\\
\rho_{\mathrm{V}}^{(\mathrm{f})}(x^0,y^0;p)\Big|_{x^0=y^0} &= 0\,, \quad & \rho_{\mathrm{T}}^{(\mathrm{f})}(x^0,y^0;p)\Big|_{x^0=y^0} &= 0\,.
\end{aligned}
$$

The initial conditions then follow for $x^0=y^0=0$.

Note that the equal-time value of the spectral function corresponds to the sum rule

$$
\int_{-\infty}^{\infty}\frac{\mathrm{d}p_0}{2\pi}\,\rho^{(\mathrm{f})}(p_0,\boldsymbol{p}) = \mathrm{i}\,\gamma^0\,. \tag{3.89}
$$

**Statistical Function**

With

$$S^{>}(x,y) = \langle \Psi(x)\overline{\Psi}(y)\rangle\,, \quad S^{<}(x,y) = -\langle \overline{\Psi}(y)\Psi(x)\rangle\,,$$

one has in thermal equilibrium [LB00]

$$\mathrm{i}\,S^{>}(p) = \Big[1 - n_{\mathrm{FD}}(p_0)\Big]\rho^{(\mathrm{f})}(p)\,, \quad \mathrm{i}\,S^{<}(p) = -n_{\mathrm{FD}}(p_0)\rho^{(\mathrm{f})}(p)\,,$$

where

$$n_{\mathrm{FD}}(p_0) = \frac{1}{\mathrm{e}^{\beta p_0} + 1} \tag{3.90}$$

is the Fermi–Dirac distribution function at inverse temperature $\beta$. Then:

$$F^{(\mathrm{f})}(p) = \frac{1}{2}\Big[S^{>}(p) + S^{<}(p)\Big] = -\,\mathrm{i}\left[\frac{1}{2} - n_{\mathrm{FD}}(p_0)\right]\rho^{(\mathrm{f})}(p)\,.$$

This is the fluctuation-dissipation relation for fermions. As for the photons, out-of-equilibrium it can be replaced by some function $\widetilde{n}^{(\mathrm{f})}$ which in general depends on $\boldsymbol{p}$ as well. Again, this does not involve any assumption. We then have:

$$F^{(\mathrm{f})}(p) = -\,\mathrm{i}\left[\frac{1}{2} - \widetilde{n}^{(\mathrm{f})}(p)\right]\rho^{(\mathrm{f})}(p)\,.$$

As for the photons, we can now derive a relation involving $\widetilde{n}^{(\mathrm{f})}$ by making use of the symmetry properties of the statistical and spectral functions. We have:

$$F^{(\mathrm{f})}(-p_0,\boldsymbol{p}) = \mathrm{i}\left[\frac{1}{2} - \widetilde{n}^{(\mathrm{f})}(p_0,\boldsymbol{p})\right]\rho^{(\mathrm{f})}(p_0,\boldsymbol{p})$$

and

$$F^{(\mathrm{f})}(-p_0,\boldsymbol{p}) = -\,\mathrm{i}\left[\frac{1}{2} - \widetilde{n}^{(\mathrm{f})}(-p_0,\boldsymbol{p})\right]\rho^{(\mathrm{f})}(-p_0,\boldsymbol{p})\,.$$

By comparison, it follows that

$$\frac{1}{2} - \widetilde{n}^{(\mathrm{f})}(p_0,\boldsymbol{p}) = -\left[\frac{1}{2} - \widetilde{n}^{(\mathrm{f})}(-p_0,\boldsymbol{p})\right] \quad \text{or} \quad 1 - \widetilde{n}^{(\mathrm{f})}(p_0,\boldsymbol{p}) - \widetilde{n}^{(\mathrm{f})}(-p_0,\boldsymbol{p}) = 0\,.$$

It can easily be checked that any solution to this equation can be parametrized as

$$\widetilde{n}^{(\mathrm{f})}(p_0,\boldsymbol{p}) = \frac{1}{\mathrm{e}^{\beta(\boldsymbol{p})} + 1}\,.$$

This function closely resembles the Fermi–Dirac distribution function (3.90), with the only difference that it does not depend on a constant temperature $\beta$ but on a "mode temperature" $\beta(\boldsymbol{p})$ which can be different for any spatial momentum $\boldsymbol{p}$.

It follows that:

$$
\begin{aligned}
&F^{(\mathrm{f})}(0,0;\boldsymbol{p}) \\
&\quad = \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, F^{(\mathrm{f})}(p_0,\boldsymbol{p}) \\
&\quad = -\,\mathrm{i} \int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi} \left[\frac{1}{2} - \widetilde{n}^{(\mathrm{f})}(p_0,\boldsymbol{p})\right] \rho^{(\mathrm{f})}(p_0,\boldsymbol{p}) \\
&\quad = -\frac{\mathrm{i}}{2}\, \underbrace{\rho^{(\mathrm{f})}(0,0;\boldsymbol{p})}_{=\mathrm{i}\,\gamma^0} + \mathrm{i} \int_{\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, \widetilde{n}^{(\mathrm{f})}(p_0,\boldsymbol{p}) \rho^{(\mathrm{f})}(p_0,\boldsymbol{p}) \\
&\quad = \frac{\gamma^0}{2} - \int_{-\infty}^{\infty} \mathrm{d}p_0 \,\mathrm{sgn}(p_0)\delta(p^2)(\gamma^0 p_0 - \boldsymbol{\gamma}\cdot\boldsymbol{p} + m^{(\mathrm{f})})\, \widetilde{n}^{(\mathrm{f})}(p_0,\boldsymbol{p}) \\
&\quad = \frac{\gamma^0}{2} - \frac{1}{2\sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}}} \int_{-\infty}^{\infty} \mathrm{d}p_0 \left[\delta\Big(p_0 - \sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}}\Big) - \delta\Big(p_0 + \sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}}\Big)\right] \\
&\qquad\qquad \cdot (\gamma^0 p_0 - \boldsymbol{\gamma}\cdot\boldsymbol{p} + m^{(\mathrm{f})})\, \widetilde{n}^{(\mathrm{f})}(p_0,\boldsymbol{p}) \\
&\quad = \frac{\gamma^0}{2} - \frac{1}{2\sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}}} \Bigg\{ \gamma^0 \sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}} \left[\widetilde{n}^{(\mathrm{f})}\Big(\sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}},\boldsymbol{p}\Big) + \widetilde{n}^{(\mathrm{f})}\Big(-\sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}},\boldsymbol{p}\Big)\right] \\
&\qquad\qquad + (-\boldsymbol{\gamma}\cdot\boldsymbol{p} + m^{(\mathrm{f})}) \left[\widetilde{n}^{(\mathrm{f})}\Big(\sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}},\boldsymbol{p}\Big) - \widetilde{n}^{(\mathrm{f})}\Big(-\sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}},\boldsymbol{p}\Big)\right] \Bigg\} \\
&\quad = \frac{\gamma^0}{2} \Bigg\{ \underbrace{1 - \left[\widetilde{n}^{(\mathrm{f})}\Big(\sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}},\boldsymbol{p}\Big) + \widetilde{n}^{(\mathrm{f})}\Big(-\sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}},\boldsymbol{p}\Big)\right]}_{=0} \Bigg\} \\
&\qquad - \frac{-\boldsymbol{\gamma}\cdot\boldsymbol{p} + m^{(\mathrm{f})}}{2\sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}}} \left[\widetilde{n}^{(\mathrm{f})}\Big(\sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}},\boldsymbol{p}\Big) - \widetilde{n}^{(\mathrm{f})}\Big(-\sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}},\boldsymbol{p}\Big)\right] \\
&\quad = \frac{-\boldsymbol{\gamma}\cdot\boldsymbol{p} + m^{(\mathrm{f})}}{\sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}}} \left[\frac{1}{2} - \widetilde{n}^{(\mathrm{f})}\Big(\sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}},\boldsymbol{p}\Big)\right] \\
&\quad = \frac{-\boldsymbol{\gamma}\cdot\boldsymbol{p} + m^{(\mathrm{f})}}{\sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}}} \left[\frac{1}{2} - n^{(\mathrm{f})}(\boldsymbol{p})\right] ,
\end{aligned}
\tag{3.91}
$$

where we have defined the fermionic nonequilibrium distribution function

$$
n^{(\mathrm{f})}(\boldsymbol{p}) = \widetilde{n}^{(\mathrm{f})}\Big(\sqrt{\boldsymbol{p}^2 + m^{(\mathrm{f})2}},\boldsymbol{p}\Big) .
$$

In terms of the Lorentz components, one then obtains:

$$
\begin{aligned}
F_{\mathrm{S}}^{(\mathrm{f})}(0,0;p) &= \frac{m^{(\mathrm{f})}}{\sqrt{p^2 + m^{(\mathrm{f})2}}} \left[\frac{1}{2} - n^{(\mathrm{f})}(p)\right] , \\
\widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(0,0;p) &= 0 , \\
F_{\mathrm{V}}^{(\mathrm{f})}(0,0;p) &= \frac{p}{\sqrt{p^2 + m^{(\mathrm{f})2}}} \left[\frac{1}{2} - n^{(\mathrm{f})}(p)\right] , \\
F_{\mathrm{T}}^{(\mathrm{f})}(0,0;p) &= 0 .
\end{aligned}
$$

# Chapter 4

# Secularities of the Equations of Motion

Having the initial conditions at hand, we will now solve the free photon EoMs analytically. Fourier transforming Eqs. (3.75a) and (3.76a) with respect to space, the free equations read:

$$\left(\frac{\partial^2}{\partial t^2} + \boldsymbol{p}^2\right) \rho^{(\mathrm{g})}_{\mu\nu}(t,t';\boldsymbol{p}) = (1-\xi)\left(\delta^0_\mu \frac{\partial}{\partial t} - \mathrm{i}\,\delta^i_\mu p_i\right) \rho^{(BA)}_\nu(t,t';\boldsymbol{p})\,, \tag{4.1a}$$

$$\left(\frac{\partial^2}{\partial t^2} + \boldsymbol{p}^2\right) \rho^{(BA)}_\mu(t,t';\boldsymbol{p}) = 0\,. \tag{4.1b}$$

The corresponding equations for the statistical function look exactly the same. In the following, the corresponding equations for the statistical function are obtained from the equations for the spectral functions by doing the following replacements:[1]

$$\begin{aligned} \sin(|\boldsymbol{p}|\,(t-t')) &\to \left[\frac{1}{2} + n^{(\mathrm{g})}(\boldsymbol{p})\right] \cos(|\boldsymbol{p}|\,(t-t'))\,, \\ \cos(|\boldsymbol{p}|\,(t-t')) &\to -\left[\frac{1}{2} + n^{(\mathrm{g})}(\boldsymbol{p})\right] \sin(|\boldsymbol{p}|\,(t-t'))\,, \end{aligned}$$

With the initial condition (3.82), we can solve the free EoM for $\rho^{(BA)}_\mu$ exactly:[2]

$$\rho^{(BA)}_\mu(t,t';\boldsymbol{p}) = \delta^0_\mu \cos(|\boldsymbol{p}|(t-t')) - \mathrm{i}\,\delta^i_\mu \frac{p_i}{|\boldsymbol{p}|} \sin(|\boldsymbol{p}|(t-t'))\,. \tag{4.2}$$

[1] This corresponds to applying

$$\left[\frac{1}{2} + n^{(\mathrm{g})}(\boldsymbol{p})\right] \frac{1}{|\boldsymbol{p}|} \frac{\partial}{\partial t}$$

to the right-hand sides of the equations.

[2] For the sake of completeness, the EoMs for the other correlation functions involving the auxiliary field are given by:

$$\left(\frac{\partial^2}{\partial t^2} + \boldsymbol{p}^2\right) \rho^{(AB)}_\mu(t,t';\boldsymbol{p}) = 0\,, \quad \left(\frac{\partial^2}{\partial t^2} + \boldsymbol{p}^2\right) \rho^{(BB)}(t,t';\boldsymbol{p}) = 0\,,$$

Plugging this back into Eq. (4.1a), we obtain:

$$\left(\frac{\partial^2}{\partial t^2}+\boldsymbol{p}^2\right)\rho^{(\mathrm{g})}_{\mu\nu}(t,t';\boldsymbol{p}) = -(1-\xi)\,|\boldsymbol{p}|\Bigg[\left(\delta^0_\mu\delta^0_\nu+\delta^i_\mu\delta^i_\nu\,\frac{p_ip_j}{\boldsymbol{p}^2}\right)\sin(|\boldsymbol{p}|(t-t'))$$
$$+\,\mathrm{i}\left(\delta^0_\mu\delta^i_\nu+\delta^i_\mu\delta^0_\nu\right)\frac{p_i}{|\boldsymbol{p}|}\cos(|\boldsymbol{p}|(t-t'))\Bigg]\,. \qquad (4.3)$$

In terms of the isotropic components, the EoMs for the spectral function read:

$$\left(\frac{\partial^2}{\partial t^2}+p^2\right)\rho^{(\mathrm{g})}_{\mathrm{S}}(t,t';p) = -(1-\xi)\,p\sin(p(t-t'))\,, \qquad (4.4\mathrm{a})$$
$$\left(\frac{\partial^2}{\partial t^2}+p^2\right)\tilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';p) = -\,\mathrm{i}(1-\xi)\,p\cos(p(t-t'))\,, \qquad (4.4\mathrm{b})$$
$$\left(\frac{\partial^2}{\partial t^2}+p^2\right)\tilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';p) = -\,\mathrm{i}(1-\xi)\,p\cos(p(t-t'))\,, \qquad (4.4\mathrm{c})$$
$$\left(\frac{\partial^2}{\partial t^2}+p^2\right)\rho^{(\mathrm{g})}_{\mathrm{T}}(t,t';p) = \quad 0\,, \qquad (4.4\mathrm{d})$$
$$\left(\frac{\partial^2}{\partial t^2}+p^2\right)\rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';p) = -(1-\xi)\,p\sin(p(t-t'))\,. \qquad (4.4\mathrm{e})$$

First of all, it has to be noted that these equations are structurally much simpler than the original ones, Eqs. (3.48) (for vanishing right-hand sides): Only second derivatives with respect to time appear, and it is obvious that the limit of Landau gauge, $\xi \to 0$, is well-defined, in contrast to the original equations, where this is not obvious. In fact, these are just equations for driven (periodically excited) harmonic oscillators with frequency $p$. In Feynman gauge, $\xi = 1$, these driving forces vanish and the EoMs become even simpler (namely those of purely harmonic oscillators which resemble the free EoMs of scalar fields, for instance). In fact, in the next section we will see that the driving forces can potentially cause problems.

If the reformulation which led to these equations were possible even for an interacting theory for a finitely truncated effective action, this formulation would seem to be the natural one for abelian gauge theories in real time. As will be shown in Chap. 5, however, the auxiliary field correlation functions are *not* free for a finitely truncated effective action, which prohibits solving them exactly and thereby "integrating them out".

and with the initial conditions (3.83) and (3.84), their solutions read:

$$\rho^{(AB)}_\mu(t,t';\boldsymbol{p}) = -\delta^0_\mu\cos(|\boldsymbol{p}|(t-t'))+\mathrm{i}\,\delta^i_\mu\,\frac{p_i}{|\boldsymbol{p}|}\sin(|\boldsymbol{p}|(t-t'))\,,\quad \rho^{(BB)}(t,t';\boldsymbol{p}) = 0\,.$$

Note that $\rho^{(AB)}_\mu(t,t';\boldsymbol{p}) = -\rho^{(BA)}_\mu(t,t';\boldsymbol{p})$, as it has to be.

## 4.1 Solution to the Free Photon Equations of Motion

The solution to the free EoM Eq. (4.3) is then easily found to be given by:

$$\begin{aligned}\rho^{(\mathrm{g})}_{0\mu\nu}(t,t';\boldsymbol{p}) = \frac{1}{|\boldsymbol{p}|}\Bigg\{ & \left[\frac{1-\xi}{2}\,|\boldsymbol{p}|(t-t')\cos(|\boldsymbol{p}|(t-t')) - \frac{1+\xi}{2}\sin(|\boldsymbol{p}|(t-t'))\right]\delta^0_\mu\delta^0_\nu \\ & + \left[\frac{1-\xi}{2}\,|\boldsymbol{p}|(t-t')\cos(|\boldsymbol{p}|(t-t')) + \frac{1+\xi}{2}\sin(|\boldsymbol{p}|(t-t'))\right]\delta^i_\mu\delta^j_\nu\,\frac{p_i p_j}{\boldsymbol{p}^2} \\ & - \mathrm{i}\,\frac{1-\xi}{2}\,|\boldsymbol{p}|(t-t')\sin(|\boldsymbol{p}|(t-t'))\left(\delta^0_\mu\delta^i_\nu + \delta^i_\mu\delta^0_\nu\right) \\ & - \sin(|\boldsymbol{p}|(t-t'))\,\delta^i_\mu\delta^j_\nu\left(g_{ij} + \frac{p_i p_j}{\boldsymbol{p}^2}\right)\Bigg\}, \end{aligned} \tag{4.5}$$

or in terms of the isotropic components:[3]

$$\rho^{(\mathrm{g})}_{0\mathrm{S}}(t,t';p) = \frac{1-\xi}{2}\,(t-t')\cos(p\,(t-t')) - \frac{1+\xi}{2}\,\frac{\sin(p\,(t-t'))}{p}, \tag{4.6a}$$

$$\widetilde{\rho}^{(\mathrm{g})}_{0\mathrm{V}}(t,t';p) = -\frac{1-\xi}{2}\,(t-t')\sin(p\,(t-t')), \tag{4.6b}$$

$$\rho^{(\mathrm{g})}_{0\mathrm{L}}(t,t';p) = -\frac{1-\xi}{2}\,(t-t')\cos(p\,(t-t')) - \frac{1+\xi}{2}\,\frac{\sin(p\,(t-t'))}{p}, \tag{4.6c}$$

$$\rho^{(\mathrm{g})}_{0\mathrm{T}}(t,t';p) = -\frac{\sin(p\,(t-t'))}{p}. \tag{4.6d}$$

Note that all components except for the spatially transverse one depend explicitly on the gauge fixing parameter and are secular, i. e. diverge in time. Only in Feynman gauge, i. e. $\xi = 1$, do the divergent terms vanish (which is already clear from looking at the reformulated EoM (4.3) and in fact even from the original one (3.18a)). This is a very peculiar result and a clear indication that those components cannot be physical since they are neither gauge invariant nor bounded in their time evolution. It is the very essence of gauge theories which shows up here, namely that there exist unphysical DoFs. From this point of view, the secularities are therefore not completely unexpected.

With the solutions (4.6) to the free EoM for the photon spectral function, we obtain for the two (potentially) physical DoFs according to Eqs. (3.41) and (3.44):

$$\rho_0^{(\mathrm{g})\,\mathrm{T}}{}_{\perp}(t,t';p) = \rho_0^{(\mathrm{g})\,\mathrm{L}}{}_{\perp}(t,t';p) = -\frac{\sin(p(t-t'))}{p}. \tag{4.7}$$

These solutions are neither secular nor do they depend on the gauge fixing parameter, as it has to be.[4] Note, however, that in vacuum (or in a system it does not interact with), the longitudinal DoF is *not* physical, as we have discussed earlier. This is because the covariant gauges are not physical gauges—they contain too many DoFs.

[3]In the free case, the two vector components are identical, so $\widetilde{\rho}^{(\mathrm{g})}_{0\mathrm{V}} := \widetilde{\rho}^{(\mathrm{g})}_{0\mathrm{V}_1} = \widetilde{\rho}^{(\mathrm{g})}_{0\mathrm{V}_2}$.

[4]In fact, up to the sign they look like the free spectral function of a massless scalar field.

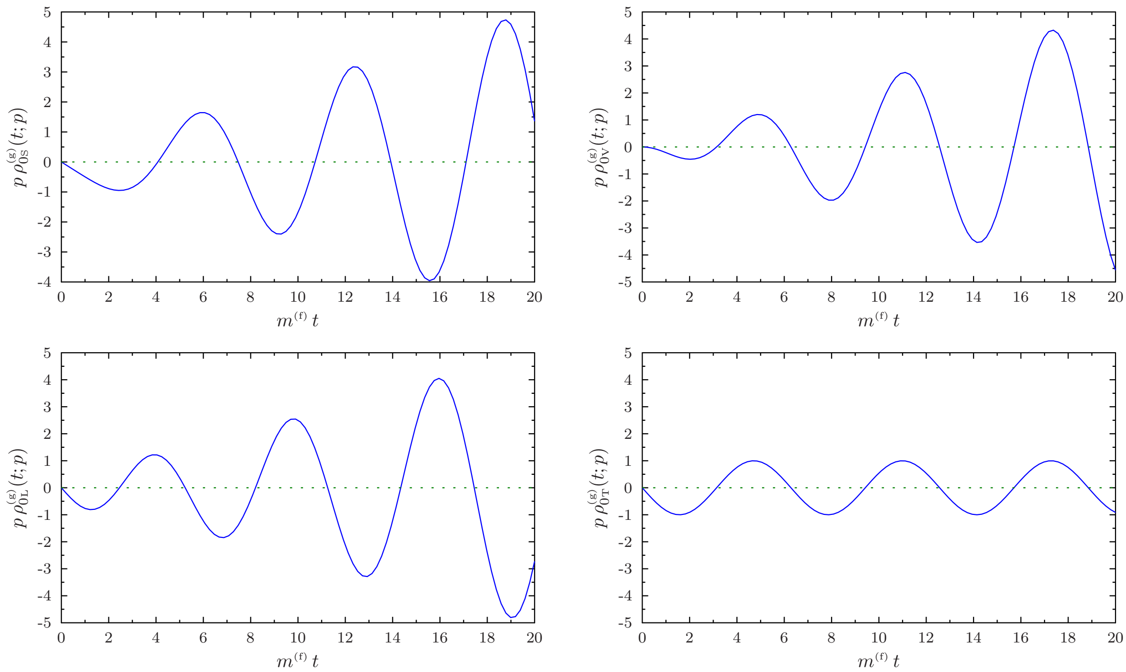


Figure 4.1: The isotropic components of the free photon spectral function for $\xi = 0.5$ and $p = 1$. Except for the transverse component, all components are secular, i. e. grow proportional to time.

## 4.2 Origin of the Secularities

There are various ways to see why the solutions (4.6) are secular in time, which are of course connected. We will discuss two of them here.

### 4.2.1 Resonance Effect

It is clear from the free EoMs (4.3) that the secularities are caused by the sine and cosine terms (and which, incidentally, are also those terms which depend explicitly on the gauge fixing parameter). The reason is that, as mentioned above, they act as a driving force in the EoMs. In fact, the free EoMs for all of the isotropic components are (for $\xi \neq 1$ and $p \neq 0$) either of the form

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + p^2\right) f(t) = \sin(p\,t) \quad \text{or} \quad \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + p^2\right) f(t) = \cos(p\,t)\,,$$

respectively. Without these terms, the isotropic components would be harmonic oscillators with frequency $p$. The sine and cosine terms, however, cause a periodic excitation with the frequency $p$. In the free case, when the memory integrals vanish, those terms therefore drive the oscillators with their eigenfrequency. The general solution to these equations is easily found to be

$$f(t) = f_0 \cos(p\,t) + \frac{\dot{f}_0}{p} \sin(p\,t) + \frac{1}{2p}\left[\frac{1}{p}\sin(p\,t) - t\cos(p\,t)\right] \quad \text{or}$$
$$f(t) = f_0 \cos(p\,t) + \frac{\dot{f}_0}{p} \sin(p\,t) + \frac{t}{2p}\sin(p\,t)\,, \tag{4.8}$$

respectively. The first two terms of each solution obviously represent the special solution, i. e. the solution to the harmonic oscillator equations, while in addition, there appears a term which is proportional to time in each case, i. e. a secular term. This secular term is independent of the initial conditions: It exists even for vanishing conditions.

The secularity of the solutions of the free photon EOM can hence be viewed as a resonance effect.

### 4.2.2 Derivative of a Delta Distribution

Another way of understanding the secularities is by considering a certain representation of the free photon spectral function in momentum-space. The usual momentum-space representation of the free photon spectral function is given by

$$\rho^{(\mathrm{g})}_{0\,\mu\nu}(p) = -2\pi\,\mathrm{i}\,\mathrm{sgn}(p_0)\delta(p^2)\left[g_{\mu\nu} - (1-\xi)\,\frac{p_\mu\,p_\nu}{p^2}\right]. \tag{4.9}$$

Employing the identity $\delta(p^2)/p^2 = -\delta'(p^2)$, another representation of the free photon spectral function in momentum space is

$$\rho^{(\mathrm{g})}_{0\,\mu\nu}(p) = -2\pi\,\mathrm{i}\,\mathrm{sgn}(p_0)\left[g_{\mu\nu}\delta(p^2) + (1-\xi)\,p_\mu\,p_\nu\,\delta'(p^2)\right]. \tag{4.10}$$

For an arbitrary function $f$ depending on $p_0$ we have:

$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}p_0\,\delta'(p^2) f(p_0) &= \int_{-\infty}^{\infty} \mathrm{d}p_0 \left\{\frac{1}{2\boldsymbol{p}^2}\,\delta(p^2) + \frac{1}{4\boldsymbol{p}^2}\left[\delta'(p_0 - |\boldsymbol{p}|) - \delta'(p_0 + |\boldsymbol{p}|)\right]\right\} f(p_0) \\
&= \int_{-\infty}^{\infty} \mathrm{d}p_0 \left\{\frac{1}{2\boldsymbol{p}^2}\,\delta(p^2) - \frac{1}{4\boldsymbol{p}^2}\left[\delta(p_0 - |\boldsymbol{p}|) - \delta(p_0 + |\boldsymbol{p}|)\right]\frac{\partial}{\partial p_0}\right\} f(p_0) \\
&= \frac{1}{2|\boldsymbol{p}|}\int_{-\infty}^{\infty} \mathrm{d}p_0\,\delta(p^2)\left[\frac{1}{|\boldsymbol{p}|} - \mathrm{sgn}(p_0)\,\frac{\partial}{\partial p_0}\right] f(p_0)\,.
\end{aligned} \tag{4.11}$$

We can already see that the partial derivative with respect to $p_0$ will, when applied to the exponential function of the Fourier transformation, yield a factor proportional to $(t - t')$, i. e. a secular term.

Using this result, the Fourier transformation of the derivative of the delta distribution is then given by:

$$
\begin{aligned}
&\int_{-\infty}^{\infty} \frac{\mathrm{d}p_0}{2\pi}\, \delta'(p^2)\ \mathrm{e}^{-\mathrm{i}\,p_0(t-t')} \\
&\quad = \frac{1}{4\pi|\boldsymbol{p}|} \int_{-\infty}^{\infty} \mathrm{d}p_0\, \delta(p^2) \left[\frac{1}{|\boldsymbol{p}|} - \mathrm{sgn}(p_0)\, \frac{\partial}{\partial p_0}\right] \mathrm{e}^{-\mathrm{i}\,p_0(t-t')} \\
&\quad = \frac{1}{8\pi \boldsymbol{p}^2} \int_{-\infty}^{\infty} \mathrm{d}p_0 \left[\delta(p_0 - |\boldsymbol{p}|) + \delta(p_0 + |\boldsymbol{p}|)\right] \left[\frac{1}{|\boldsymbol{p}|} + \mathrm{i}\,\mathrm{sgn}(p_0)(t-t')\right] \mathrm{e}^{-\mathrm{i}\,p_0(t-t')} \\
&\quad = \frac{1}{8\pi|\boldsymbol{p}|^3} \int_{-\infty}^{\infty} \mathrm{d}p_0 \Big\{ \delta(p_0 - |\boldsymbol{p}|) + \delta(p_0 + |\boldsymbol{p}|) \\
&\qquad\qquad + \mathrm{i}|\boldsymbol{p}|(t-t') \left[\delta(p_0 - |\boldsymbol{p}|) - \delta(p_0 + |\boldsymbol{p}|)\right] \Big\}\, \mathrm{e}^{-\mathrm{i}\,p_0(t-t')} \\
&\quad = \frac{1}{4\pi|\boldsymbol{p}|^3} \left[\cos(|\boldsymbol{p}|(t-t')) + |\boldsymbol{p}|(t-t')\sin(|\boldsymbol{p}|(t-t'))\right].
\end{aligned}
\tag{4.12}
$$

So it is obviously the $\delta'(p^2)$-term which causes the secularity, and which comes from the $(1-\xi)\delta(p^2)p_\mu p_\nu/p^2$-term in the free spectral function. This is also another way to see why there are no secular terms in Feynman gauge, i. e. $\xi = 1$.

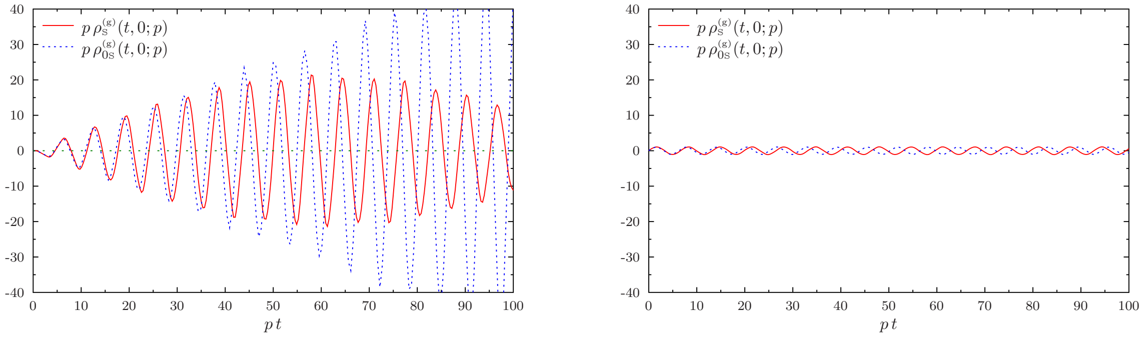
### 4.2.3 Secularities and the Full Theory

It is an important question if the secularities persist in the full theory. Even if they did, though, this would not indicate a failure of the theory or its formulation in real time; after all, the secular components are not physical and hence not observable.[5] It would certainly impose practical complications, however. If no further approximations are made, the only way to treat the EoMs is by means of numerical methods. Numerically handling large, diverging quantities is delicate, however, in particular if one is interested in differences of diverging quantities which may be finite in an exact calculation.

The question if the secularities persist in the full theory is not easy to answer. Since QED is a very weakly coupled theory, significant deviations from the free solutions are to be expected at rather late times only, and reaching late times with a numerical simulation is challenging due to the required memory resources. It is, however, rather likely that the secularities do *not* persist in the full theory. Due to the presence of the memory integrals, the frequency of the driving terms will be slightly shifted away from $p$, thereby destroying the resonance which, as we have seen in a previous section, is responsible for the secularities.

In a very rough approximation, one may set the memory integrals to a constant value (which is small since it is proportional to the squared coupling) times the spectral function (so as to keep the self-consistent nature of the equations, but neglecting the nonlocality in time as well as the coupling to the other isotropic components). For instance, for the

[5] Any finite quantity can be split into diverging parts. As a trivial example, consider the function $f(x) = 0$ which is zero identically. We can certainly write $f(x) = g(x) + h(x)$ with $g(x) = x$ and $h(x) = -x$. Both $g$ and $h$ diverge, but their sum remains finite.




(a) Landau gauge ($\xi = 0$) (b) Feynman gauge ($\xi = 1$)

Figure 4.2: The scalar component of the (maximally) unequal-time photon spectral function in the (very rough) approximation (4.14). As an example for a solution with gauge fixing parameter $\xi \neq 1$ and which therefore is secular in the free case, the plot on the left-hand side shows the Landau gauge solution ($\xi = 0$). Although it remains finite at all times, its maximum value is still very large compared to the Feynman gauge solution shown in the plot on the right-hand side (where only the frequency of the oscillations becomes shifted, not the amplitude).

scalar component, we then have $I^{\rm (g)}_{(\rho)\rm S}(t,t';p) \approx e^2p^2 f(p)\,\rho_{\rm S}^{\rm (g)}(t,t';p)$ for some function $f(p)$. Then the equation for the scalar component reads:

$$\left\{\frac{\partial^2}{\partial t^2} + \Big[1 - e^2 f(p)\Big]\,p^2\right\}\rho_{\rm S}^{\rm (g)}(t,t';p) = -(1-\xi)\,p\sin(p\,(t-t'))\,, \tag{4.13}$$

corresponding to a oscillation frequency shift

$$p \to p' = \sqrt{1-e^2 f(p)}\,p = \left[1 - \frac{e^2 f(p)}{2}\right]p + \mathcal{O}(e^4)\,,$$

i. e. the difference in the frequency is $\Delta p = |p'-p| = e^2 f(p)p/2 + \mathcal{O}(e^4)$. Its solution is given by:

$$\begin{aligned}\rho_{\rm S}^{\rm (g)}(t,t';p) &= (1-\xi)\,\frac{\sqrt{1-e^2f(p)}\,\sin(p\,(t-t')) - \sin\!\Big(\sqrt{1-e^2f(p)}\,p\,(t-t')\Big)}{e^2 f(p)\,\sqrt{1-e^2 f(p)}\,p} \\ &\quad - \xi\,\frac{\sin\!\Big(\sqrt{1-e^2f(p)}\,p\,(t-t')\Big)}{\sqrt{1-e^2f(p)}\,p} \\ &= \frac{p^2}{p^2-p'^2}\left[(1-\xi)\,\frac{\sin(p\,(t-t'))}{p} - \left(1-\frac{p'^2}{p^2}\,\xi\right)\frac{\sin(p'\,(t-t'))}{p'}\right]\,. \end{aligned} \tag{4.14}$$

Obviously, the secular term appearing in the solution to the corresponding free EOM has vanished, and the solution remains finite at all times.[6]

[6]For gauge fixing parameters not too close to unity, the solution corresponds to a beat with frequency

Expanding the solution in the squared coupling yields

$$\begin{aligned}\rho_{\mathrm{S}}^{(\mathrm{g})}(t,t';p) = \rho_{0\mathrm{S}}^{(\mathrm{g})}(t,t';p) + \frac{e^2 f(p)}{8}\bigg\{ & (3+\xi)\left[(t-t')\cos(p\,(t-t')) - \frac{\sin(p\,(t-t'))}{p}\right] \\ & + (1-\xi)\,p^2\,(t-t')^2\,\frac{\sin(p\,(t-t'))}{p}\bigg\} \\ & + \mathcal{O}(e^4)\,.\end{aligned}$$

Note that due to the self-consistent nature of the equation, we do not obtain an expansion in $e^2\,p\,(t-t')$, as we would had we expanded $\rho_{\mathrm{S}}^{(\mathrm{g})}(t,t';p)$ in the right-hand side of the EoM in the coupling as well, but only in $e^2$. However, expanding the solution to any finite order again yields secular solutions.

So far, however, we have not observed this behavior numerically. In fact, our results indicate that at early times, the solutions to the full theory actually grow even stronger than the free ones. It has to be stressed, however, that this is not a counterargument to our reasoning above: It may well be that there is a phase of increased growth before the damping due to the shift of the resonance frequency rendering the solutions finite sets in. Due to the complicated structure and nonlocality with respect to time of the memory integrals and the coupling of the different isotropic components with each other, our above analytical estimate of course falls short of capturing many aspects of the full theory.

It should be mentioned that non-Gaussian initial conditions might also lead to nonsecular solutions. This is because, as described earlier, non-Gaussian initial conditions manifest themselves as higher-order interaction terms which only act at initial time. However, this might be sufficient to kick the solutions away from the resonance, thereby avoiding the secularities.

one half times the frequency shift, i. e. the solution is in particular periodic. This is of course not realistic (in physical situations, one expects damping), but shows that a shift in the frequency leads to finite solutions.

# Chapter 5

# Gauge Invariance and the Ward Identities in the 2PI Framework

Most problems in formulations of gauge theories derived from the 2PI effective action are related to gauge invariance and stem from the resummation implemented by the 2PI effective action which mixes up different perturbative orders.[1] This is why we will start this chapter by discussing this peculiar feature of the 2PI effective action. Two important issues we will then discuss are the applicability of Ward identities to correlation functions derived from the 2PI effective action, and the dependence of gauge invariant quantities on the gauge fixing parameter.

We will use a slightly different (or rather more precise) notation in this chapter than in the rest of the work. This is because it is important here to clearly distinguish various related objects, e. g. fields appearing as variational parameters of the effective action and their physical values (denoted by an underscore in this chapter) or correlation functions obtained from the 2PI effective action and from the 2PI-resummed effective action (to be defined later in this chapter). In order not to completely mess up the notation, we will dispense with also distinguishing between quantities derived from the exact effective action and from a truncated one; which one is meant will become clear from the context.

## 5.1 Resummation and the Mixing of Perturbative Orders

Each diagram contained in the 2PI part of the 2PI effective action resums an infinite number of perturbative diagrams, and in its expansion perturbative diagrams of every order appear. It is then interesting to compare the expansion of a finite truncation of the 2PI effective action in terms of perturbative diagrams with the (a priori perturbative) expansion of the 1PI effective action to the same order. The result is always that the

[1] A closely related problem is the validity of the Goldstone theorem [Gol61] in scalar theories with a spontaneously broken continuous symmetry [BG77, IRK05].

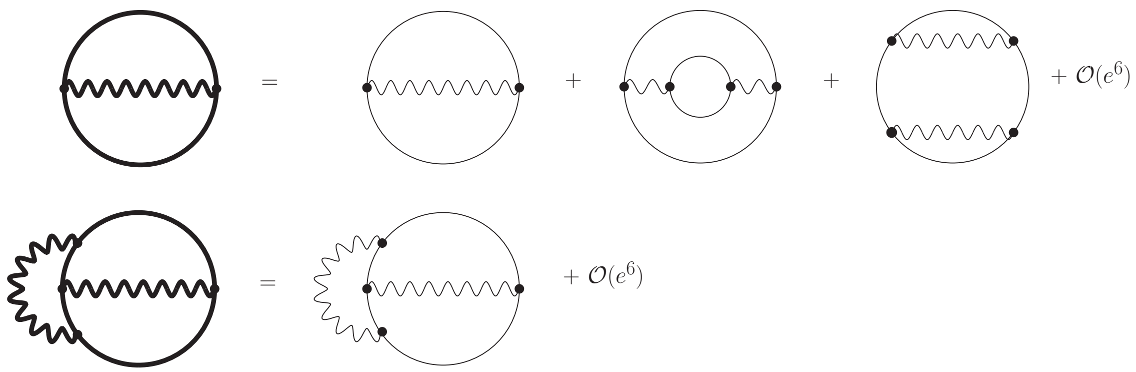



Figure 5.1: The perturbative expansion (i. e. expansion in free propagators) of the two- and three-loop contributions to the 2PI effective action.

perturbative expansion of the 1PI effective action and the 2PI effective action to any given order do *not* agree: The 1PI effective action expansion to a given order always contains more diagrams than the 2PI expansion. The reason is simply that the expansion of the 1PI effective action contains *every* diagram up to that order (by definition), while only diagrams appear in the perturbative expansion of the 2PI effective action which can be generated by expanding propagators *only* in each 2PI diagram (since vertices are bare).

In order to illustrate this, let us consider the two-loop truncation of the 2PI effective action which we will use for the numerics (see the next chapter), and let us compare its perturbative expansion up to three loops with the expansion of the 1PI effective action up to three loops. There are two two-loop diagrams in the expansion of the 2PI effective action as well as in the expansion of the 1PI effective action, namely the first two diagrams on the right-hand side of the upper part of Fig. (5.1). We conclude that their expansions up to two loops agree. However, there is only *one* three-loop diagram in the perturbative expansion of the 2PI effective action (the third one on the right-hand side of the upper part of Fig. (5.1)), while there is an additional one in the expansion of the 1PI effective action. This additional diagrams corresponds to a vertex dressing, and it is clear that one cannot obtain a vertex dressing by expanding propagators only as in the 2PI effective action. Therefore, the difference in the perturbative expansions of the 1PI effective action and of the 2PI effective action is precisely given by the lowest vertex dressing diagram. This diagram only appears in the perturbative expansion of the three-loop truncation of the 2PI effective action, as shown in the lower part of Fig. (5.1). It is then also clear that in the exact theory the expansions coincide.

Continuing along the same lines, one would find that the perturbative expansions of the 1PI effective action and of the three-loop truncation of the 2PI effective action would agree up to three loops, while differences would appear at four loops. It can be shown that in general, the perturbative expansions of the 1PI effective action and of the $n$-loop truncated 2PI effective action agree up to $n$ loops [AAB+02, vHK02, BBRS05].

## 5.2 The Ward Identities

One important characteristic property of gauge theories is that there exist identities between correlation functions of different order, the so-called *Ward identities*[2] [Tak57, War50]. In approaches based on an $n$PI effective action, this is potentially problematic since all correlation functions up to the $n$th order are full, while all higher-order correlation functions are free. A simple example is the one-loop photon self-energy (6.4): It depends on two *full* fermion propagators, but also on two *bare* vertices.

Problems regarding the Ward identities and the 2PI effective action have been discussed e. g. in Refs. [Mot03, CKZ05, Cal04, AS02].

### 5.2.1 Standard Ward Identities

The standard Ward identities for the 1PI effective action can be obtained from the master equation[3]

$$\mathcal{G}(x)\Big(\Gamma_{1\mathrm{PI}}[A,\overline{\psi},\psi] - S^{\xi}_{\mathrm{gf}}[A]\Big) = 0\,, \tag{5.1}$$

with the generator of gauge transformations (2.12). This identity states that the (1PI) effective action is gauge invariant up to the gauge fixing term (which, after all, has been introduced to break gauge invariance in the first place). The infinite tower of Ward identities can then be generated by Taylor-expanding the master Ward identity, i. e. by taking derivatives of this identity with respect to the fields and evaluating the resulting equation for their physical values obtained by solving their EoMs[4]

$$\left.\frac{\delta\Gamma_{1\mathrm{PI}}[A,\overline{\psi},\psi]}{\delta A_\mu(x)}\right|_{\mathrm{phys}} = 0\,, \quad \left.\frac{\delta\Gamma_{1\mathrm{PI}}[A,\overline{\psi},\psi]}{\delta\overline{\psi}(x)}\right|_{\mathrm{phys}} = 0\,, \quad \left.\frac{\delta\Gamma_{1\mathrm{PI}}[A,\overline{\psi},\psi]}{\delta\psi(x)}\right|_{\mathrm{phys}} = 0\,, \tag{5.2}$$

so that

$$\frac{\delta^k}{\delta A_{\mu_1}(x_1)\ldots\delta A_{\mu_k}(x_k)}\frac{\delta^l}{\delta\overline{\psi}(x_{k+1})\ldots\delta\overline{\psi}(x_{k+l})}\frac{\delta^m}{\delta\psi(x_{k+l+1})\ldots\psi(x_{k+l+m})}$$
$$\mathcal{G}(x)\Big(\Gamma_{1\mathrm{PI}}[A,\overline{\psi},\psi] - S^{\xi}_{\mathrm{gf}}[A]\Big)\Big|_{\mathrm{phys}} = 0\,. \tag{5.3}$$

Variations of the 1PI effective action with respect to fields evaluated at their physical values correspond to (1PI) correlation functions. For instance, the (full) inverse photon

[2] Also called *Ward–Takahashi identities*; we will, however, stick to the shorter "Ward identities".

[3] A derivation of this equation is rather easy and can be found in most introductory textbooks on QFT, e. g. [Ryd96]. We only outline one possible derivation here: One starts by performing a gauge transformation on the generating functional of correlation functions $Z$. The requirement of $Z$ being gauge invariant yields a condition which can be expressed as a functional differential equation for $Z$, which can then be translated into an equation for the generating functional of connected correlation functions $W$. By performing a Legendre transform, this equation can in turn be translated into an equation for the generating functional of proper vertex functions, i. e. the effective action. This equation is just the one we state in the main text.

[4] By $\ldots|_{\mathrm{phys}}$ we mean those values of the arguments of the given quantity which solve its EoM.

and fermion propagators are defined as

$$\mathrm{i}\,\underline{S}^{-1}(x,y) = \left.\frac{\delta^2\Gamma_{1\mathrm{PI}}[A,\overline{\psi},\psi]}{\delta\psi(x)\delta\overline{\psi}(y)}\right|_{\mathrm{phys}}\,, \quad \mathrm{i}\left(\underline{D}^{-1}\right)^{\mu\nu}(x,y) = \left.\frac{\delta^2\Gamma_{1\mathrm{PI}}[A,\overline{\psi},\psi]}{\delta A_\mu(x)\delta A_\nu(y)}\right|_{\mathrm{phys}}\,, \tag{5.4}$$

Similarly, the full electron-photon vertex is defined as

$$\mathrm{i}\,\underline{\Gamma}^\mu(x,y,z) = \left.\frac{\delta^3\Gamma_{1\mathrm{PI}}[A,\overline{\psi},\psi]}{\delta A_\mu(x)\delta\overline{\psi}(y)\delta\psi(z)}\right|_{\mathrm{phys}}\,, \tag{5.5}$$

and so on for higher correlation functions.

Except for the case $k = l = m = 0$, which yields a trivial identity which is true due to the EoMs for the one-point functions, one therefore obtains nontrivial identities between correlation functions. In particular, for the case $k \neq 0$, $l = m = 0$, one obtains

$$\partial_{x^\mu} \frac{\delta^k}{\delta A_{\mu_1}(x_1)\ldots A_{\mu_k}(x_k)}\left(\Gamma_{1\mathrm{PI}}[A,\overline{\psi},\psi] - S^\xi_{\mathrm{gf}}[A]\right)\Big|_{\mathrm{phys}} = 0\,, \tag{5.6}$$

i. e. the statement that the longitudinal part of all photon $n$-point functions is not modified by quantum corrections. For the case $k = 2$, one obtains the important result that the longitudinal part of the photon self-energy vanishes identically.

By truncating the effective action at any finite order in a perturbative expansion, it follows that the Ward identities hold at each perturbative order.

### 5.2.2 Ward Identities in the 2PI Framework

For the 2PI effective action, however, things are more complicated since the 2PI effective action depends on a larger number of parameters (the field expectation values *and* the propagators). It is therefore obvious that one cannot simply translate the standard 1PI Ward identities to corresponding identities involving the 2PI effective action.

A master equation very similar to Eq. (5.1) can be written down for the 2PI effective action as well.[5] It is given by:

$$\mathcal{G}_{2\mathrm{PI}}(x,y)\left(\Gamma_{2\mathrm{PI}}[A,\overline{\psi},\psi,D,S] - S^\xi_{\mathrm{gf}}[A]\right) = 0\,, \tag{5.7}$$

[5] In fact, it can be shown that there is an object derived from the 2PI effective action for which the standard Ward identities do hold [BS04, RS07] (similarly, propagators in a theory with a spontaneously broken continuous symmetry derived from this object do exhibit a Goldstone mode [AAB+02, vHK02]): The so-called 2PI-*resummed* effective action, defined by

$$\Gamma^{\mathrm{res}}_{2\mathrm{PI}}[A,\overline{\psi},\psi] = \Gamma_{2\mathrm{PI}}[A,\overline{\psi},\psi,\underline{D}(A,\overline{\psi},\psi),\underline{S}(A,\overline{\psi},\psi)]\,,$$

where $\underline{D}(A,\overline{\psi},\psi)$ and $\underline{S}(A,\overline{\psi},\psi)$ are the physical values of the propagators, which are in turn defined by

$$\left.\frac{\delta\Gamma_{2\mathrm{PI}}[A,\overline{\psi},\psi,D,S]}{\delta D_{\mu\nu}(x,y)}\right|_{D_{\mu\nu}=\underline{D}_{\mu\nu},S=\underline{S}} = 0\,, \quad \left.\frac{\delta\Gamma_{2\mathrm{PI}}[A,\overline{\psi},\psi,D,S]}{\delta S(x,y)}\right|_{D_{\mu\nu}=\underline{D}_{\mu\nu},S=\underline{S}} = 0\,.$$

The 2PI-resummed effective action depends on the same arguments as the 1PI effective action, i. e. on the field expectation values. It should be noted, however, that for a given truncation of the effective action, where the generator of gauge transformation now reads:

$$\mathcal{G}_{2\text{PI}}(x,y) = \delta^4(x-y)\,\mathcal{G}(x) + \mathrm{i}\left[S(x,y)\,\frac{\delta}{\delta S(x,y)} - S(y,x)\,\frac{\delta}{\delta S(y,x)}\right]. \tag{5.8}$$

If the 2PI effective action is independent of the one-point functions, this simplifies to

$$\left[S(x,y)\,\frac{\delta}{\delta S(x,y)} - S(y,x)\,\frac{\delta}{\delta S(y,x)}\right]\Gamma_{2\text{PI}}[D,S] = 0\,. \tag{5.9}$$

It is immediately clear that this cannot constrain the photon self-energy in any way. As for the standard (1PI) Ward identities, the simplest identity which can be generated from the above master Ward identity is obtained by taking the derivative with respect to one of the propagators. This, however, generates an identity for four-point functions.

In the next section, we will discuss the nontransversality of the photon self-energy (3.8).

### 5.2.3 Non-Transversality of the Photon Self-Energy

Since we are only concerned with two-point functions in this work, of interest for us is mainly the Ward identity stating the transversality of the photon self-energy. Above, we have shown that the longitudinal part of all photon $n$-point functions is not modified by quantum corrections. We have for the 1PI effective action:

$$\frac{\delta}{\delta A_\nu(y)}\left\{\mathcal{G}(x)\Big(\Gamma_{1\text{PI}}[A,\psi,\overline{\psi}] - S^{\xi}_{\text{gf}}[A]\Big)\right\}\Big|_{\text{phys}} = 0\,, \tag{5.10}$$

or explicitly:

$$\begin{aligned}
0 &= \mathrm{i}\;\partial_{x\mu}\,\frac{\delta^2}{\delta A_\mu(x)\delta A_\nu(y)}\Big(\Gamma_{1\text{PI}}[A,\overline{\psi},\psi] - S^{\xi}_{\text{gf}}[A]\Big)\Big|_{\text{phys}} \\
&= \mathrm{i}\;\partial_{x\mu}\left\{\frac{\delta^2\Gamma_{1\text{PI}}[A,\overline{\psi},\psi]}{\delta A_\mu(x)\delta A_\nu(y)} - \frac{1}{\xi}\,\partial^\mu_x\partial^\nu_x\delta^4(x-y)\right\}\Big|_{\text{phys}} \\
&= -\partial_{x\mu}\left[\Big(\underline{D}^{-1}\Big)^{\mu\nu}(x,y) - \Big(D_0^{-1}\Big)^{\mu\nu}(x,y)\right]\Big|_{\text{phys}} \\
&= \partial_{x\mu}\widetilde{\Pi}^{\mu\nu}(x,y)\,,
\end{aligned} \tag{5.11}$$

where we have used that the physical values of the field expectation values vanish.

---

the resummed 2PI effective action contains much more information than the 1PI effective action since it is constructed using full (with respect to the given truncation) propagators, whereas the 1PI effective action is constructed using free propagators.

The Ward identities then read:

$$\left\{\mathcal{G}(x)\big(\Gamma^{\text{res}}_{2\text{PI}}[A,\overline{\psi},\psi] - S^{\xi}_{\text{gf}}[A]\big)\right\}\Big|_{A_\mu=\underline{A}_\mu,\overline{\psi}=\underline{\overline{\psi}},\psi=\underline{\psi}} = 0\,.$$

The important point now is that for a finitely truncated 2PI effective action, the photon self-energy defined by

$$\Pi^{\mu\nu}(x,y) = 2\,\mathrm{i}\,\frac{\delta\Gamma_2[A,\psi,\overline{\psi},D,S]}{\delta D_{\mu\nu}(x,y)}\bigg|_{\mathrm{phys}} \tag{5.12}$$

is in general *not* transverse, since for any finite truncation of $\Gamma_{2\mathrm{PI}}[D,S]$, we have $\widetilde{\Pi}^{\mu\nu}(x,y) \neq \Pi^{\mu\nu}(x,y)$.

Similar statements can be made for higher-order correlation functions. It should be stressed, however, that the failure of the photon self-energy (and of all other photon correlation functions) defined as the variation of the 2PI part of the 2PI effective action to be transverse does *not* imply that the Ward identities are violated, neither does it imply a failure of the theory. It simply means that since in a finitely truncated theory, different definitions of correlation function do not have to coincide, they can be constrained in different ways (or even not at all).

It should also be noted that there exist relations between the variational and the 2PI-resummed propagators. In fact, the 2PI-resummed propagators satisfy a Dyson–Schwinger-like equation including the variational propagators [RS07].

Finally, it is instructive to see explicitly that the one-loop photon self-energy as derived from the 2PI effective action is not transverse in general, which is what we will turn to next.

### Non-Transversality of the One-Loop Photon Self-Energy

The nontransversality of the one-loop photon self-energy can seen as follows: First one decomposes the self-energy into statistical and spectral parts. The partial derivative acting on the photon self-energy effectively acts on the fermion propagators, and their EoMs (3.20) can be used to replace the resulting gradients. Since the full fermion propagator is the sum of the free propagator plus a correction due to the interaction and the photon self-energy constructed from free fermion propagators is transverse, we are left with a term which is due to interactions only. This can be shown as follows:

First of all note that the statistical and spectral photon self-energies are real. Hermiticity implies that [BBS03] (see also App. D)

$$F^{(\mathrm{f})}(y,x) = \quad \gamma^0 F^{(\mathrm{f})}(x,y)^\dagger \gamma^0\,, \quad \rho^{(\mathrm{f})}(y,x) = -\gamma^0 \rho^{(\mathrm{f})}(x,y)^\dagger \gamma^0\,.$$

Then:

$$\begin{aligned}
\Pi^{\mu\nu}_{(F)}(x,y)^* &= e^2\,\mathrm{tr}\Big(\gamma^\mu F^{(\mathrm{f})}(x,y)\gamma^\nu F^{(\mathrm{f})}(y,x) + \frac{1}{4}\,\gamma^\mu \rho^{(\mathrm{f})}(x,y)\gamma^\nu \rho^{(\mathrm{f})}(y,x)\Big)^* \\
&= e^2\,\mathrm{tr}\Big(\Big[\gamma^\mu F^{(\mathrm{f})}(x,y)\gamma^\nu F^{(\mathrm{f})}(y,x) + \frac{1}{4}\,\gamma^\mu \rho^{(\mathrm{f})}(x,y)\gamma^\nu \rho^{(\mathrm{f})}(y,x)\Big]^\dagger\Big) \\
&= e^2\,\mathrm{tr}\Big(F^{(\mathrm{f})}(y,x)^\dagger(\gamma^\nu)^\dagger F^{(\mathrm{f})}(x,y)^\dagger(\gamma^\mu)^\dagger + \frac{1}{4}\,\rho^{(\mathrm{f})}(y,x)^\dagger(\gamma^\nu)^\dagger \rho^{(\mathrm{f})}(x,y)^\dagger(\gamma^\mu)^\dagger\Big) \\
&= e^2\,\mathrm{tr}\Big(\gamma^0 F^{(\mathrm{f})}(x,y)\gamma^0\gamma^0\gamma^\nu\gamma^0\gamma^0 F^{(\mathrm{f})}(y,x)\gamma^0\gamma^0\gamma^\mu\gamma^0
\end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{4}\gamma^0\rho^{(\mathrm{f})}(x,y)\gamma^0\gamma^0\gamma^\nu\gamma^0\gamma^0\rho^{(\mathrm{f})}(y,x)\gamma^0\gamma^0\gamma^\mu\gamma^0\Big) \\
&= e^2\,\mathrm{tr}\Big(\gamma^\mu F^{(\mathrm{f})}(x,y)\gamma^\nu F^{(\mathrm{f})}(y,x) + \frac{1}{4}\gamma^\mu\rho^{(\mathrm{f})}(x,y)\gamma^\nu\rho^{(\mathrm{f})}(y,x)\Big) \\
&= \Pi^{\mu\nu}_{(F)}(x,y)\,,
\end{aligned}
$$

where we have used that $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$ and $(\gamma^0)^2 = \mathbf{1}$, i. e. $\Pi^{\mu\nu}_{(F)}(x,y)$ is real. By a similar calculation, one finds that the spectral part $\Pi^{\mu\nu}_{(\rho)}(x,y)$ is real as well. Next, from the EoMs for the fermion spectral and statistical functions (see Eqs. (3.20)),

$$
\begin{aligned}
\Big(\mathrm{i}\,\gamma^\mu\partial_{x\mu} - m^{(\mathrm{f})}\Big)\rho^{(\mathrm{f})}(x,y) &= I^{(\mathrm{f})}_{(\rho)}(x,y)\,, \\
\Big(\mathrm{i}\,\gamma^\mu\partial_{x\mu} - m^{(\mathrm{f})}\Big)F^{(\mathrm{f})}(x,y) &= I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)\,,
\end{aligned}
$$

where $I^{(\mathrm{f})}_{(\rho)}$ and $I^{(\mathrm{f})}_{(\mathrm{F})}$ are the corresponding memory integrals, it then follows that:

$$
\begin{aligned}
\gamma^\mu\partial_{x\mu}\rho^{(\mathrm{f})}(x,y) &= -\,\mathrm{i}\Big[m^{(\mathrm{f})}\rho^{(\mathrm{f})}(x,y) + I^{(\mathrm{f})}_{(\rho)}(x,y)\Big]\,, \\
\gamma^\mu\partial_{x\mu}F^{(\mathrm{f})}(x,y) &= -\,\mathrm{i}\Big[m^{(\mathrm{f})}F^{(\mathrm{f})}(x,y) + I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)\Big]\,,
\end{aligned}
$$

and from the hermitean conjugate EoMs,

$$
\begin{aligned}
\partial_{x\mu}\rho^{(\mathrm{f})}(y,x)\gamma^\mu &= \mathrm{i}\Big[m^{(\mathrm{f})}\rho^{(\mathrm{f})}(y,x) - \gamma^0 I^{(\mathrm{f})}_{(\rho)}(x,y)^\dagger\gamma^0\Big]\,, \\
\partial_{x\mu}F^{(\mathrm{f})}(y,x)\gamma^\mu &= \mathrm{i}\Big[m^{(\mathrm{f})}F^{(\mathrm{f})}(y,x) + \gamma^0 I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)^\dagger\gamma^0\Big]\,.
\end{aligned}
$$

Then:

$$
\begin{aligned}
\partial_{x\mu}\Pi^{\mu\nu}_{(F)}(x,y) = e^2\,\mathrm{tr}\Big(&\quad\gamma^\mu\Big[\partial_{x\mu}F^{(\mathrm{f})}(x,y)\Big]\gamma^\nu F^{(\mathrm{f})}(y,x) + \gamma^\mu F^{(\mathrm{f})}(x,y)\gamma^\nu\partial_{x\mu}F^{(\mathrm{f})}(y,x) \\
&+ \frac{1}{4}\Big\{\gamma^\mu\Big[\partial_{x\mu}\rho^{(\mathrm{f})}(x,y)\Big]\gamma^\nu\rho^{(\mathrm{f})}(y,x) + \gamma^\mu\rho^{(\mathrm{f})}(x,y)\gamma^\nu\partial_{x\mu}\rho^{(\mathrm{f})}(y,x)\Big\}\Big) \\
= e^2\,\mathrm{tr}\Big(&\quad -\,\mathrm{i}\Big[m^{(\mathrm{f})}F^{(\mathrm{f})}(x,y) + I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)\Big]\gamma^\nu F^{(\mathrm{f})}(y,x) \\
&\quad + \mathrm{i}\,F^{(\mathrm{f})}(x,y)\gamma^\nu\Big[m^{(\mathrm{f})}F^{(\mathrm{f})}(y,x) + \gamma^0 I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)^\dagger\gamma^0\Big] \\
&+ \frac{1}{4}\Big\{-\,\mathrm{i}\Big[m^{(\mathrm{f})}\rho^{(\mathrm{f})}(x,y) + I^{(\mathrm{f})}_{(\rho)}(x,y)\Big]\gamma^\nu\rho^{(\mathrm{f})}(y,x) \\
&\quad + \mathrm{i}\,\rho^{(\mathrm{f})}(x,y)\gamma^\nu\Big[m^{(\mathrm{f})}\rho^{(\mathrm{f})}(y,x) - \gamma^0 I^{(\mathrm{f})}_{(\rho)}(x,y)^\dagger\gamma^0\Big]\Big\}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \mathrm{i}\, e^2 \,\mathrm{tr}\Bigg( \qquad - m^{(\mathrm{f})} F^{(\mathrm{f})}(x,y)\gamma^\nu F^{(\mathrm{f})}(y,x) - I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)\gamma^\nu F^{(\mathrm{f})}(y,x) \\
&\qquad\qquad + m^{(\mathrm{f})} F^{(\mathrm{f})}(x,y)\gamma^\nu F^{(\mathrm{f})}(y,x) + F^{(\mathrm{f})}(x,y)\gamma^\nu \gamma^0 I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)^\dagger \gamma^0 \\
&\qquad + \frac{1}{4}\Big\{ - m^{(\mathrm{f})} \rho^{(\mathrm{f})}(x,y)\gamma^\nu \rho^{(\mathrm{f})}(y,x) - I^{(\mathrm{f})}_{(\rho)}(x,y)\gamma^\nu \rho^{(\mathrm{f})}(y,x) \\
&\qquad\qquad + m^{(\mathrm{f})} \rho^{(\mathrm{f})}(x,y)\gamma^\nu \rho^{(\mathrm{f})}(y,x) - \rho^{(\mathrm{f})}(x,y)\gamma^\nu \gamma^0 I^{(\mathrm{f})}_{(\rho)}(x,y)^\dagger \gamma^0 \Big\}\Bigg) \\
&= -\mathrm{i}\, e^2 \,\mathrm{tr}\Bigg( \qquad I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)\gamma^\nu F^{(\mathrm{f})}(y,x) - F^{(\mathrm{f})}(x,y)\gamma^\nu \gamma^0 I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)^\dagger \gamma^0 \\
&\qquad + \frac{1}{4}\Big[ I^{(\mathrm{f})}_{(\rho)}(x,y)\gamma^\nu \rho^{(\mathrm{f})}(y,x) + \rho^{(\mathrm{f})}(x,y)\gamma^\nu \gamma^0 I^{(\mathrm{f})}_{(\rho)}(x,y)^\dagger \gamma^0 \Big]\Bigg) \\
&= -\mathrm{i}\, e^2 \,\mathrm{tr}\Bigg( \qquad I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)\gamma^\nu F^{(\mathrm{f})}(y,x) - \gamma^0 F^{(\mathrm{f})}(y,x)^\dagger \gamma^0 \gamma^\nu \gamma^0 I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)^\dagger \gamma^0 \\
&\qquad + \frac{1}{4}\Big[ I^{(\mathrm{f})}_{(\rho)}(x,y)\gamma^\nu \rho^{(\mathrm{f})}(y,x) - \gamma^0 \rho^{(\mathrm{f})}(y,x)^\dagger \gamma^0 \gamma^\nu \gamma^0 I^{(\mathrm{f})}_{(\rho)}(x,y)^\dagger \gamma^0 \Big]\Bigg) \\
&= -\mathrm{i}\, e^2 \,\mathrm{tr}\Bigg( \qquad I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)\gamma^\nu F^{(\mathrm{f})}(y,x) - F^{(\mathrm{f})}(y,x)^\dagger \Big(\gamma^\nu\Big)^\dagger I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)^\dagger \\
&\qquad + \frac{1}{4}\Big[ I^{(\mathrm{f})}_{(\rho)}(x,y)\gamma^\nu \rho^{(\mathrm{f})}(y,x) - \rho^{(\mathrm{f})}(y,x)^\dagger \Big(\gamma^\nu\Big)^\dagger I^{(\mathrm{f})}_{(\rho)}(x,y)^\dagger \Big]\Bigg) \\
&= -\mathrm{i}\, e^2 \,\mathrm{tr}\Bigg( \qquad I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)\gamma^\nu F^{(\mathrm{f})}(y,x) - \Big[ I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)\gamma^\nu F^{(\mathrm{f})}(y,x)\Big]^\dagger \\
&\qquad + \frac{1}{4}\Big\{ I^{(\mathrm{f})}_{(\rho)}(x,y)\gamma^\nu \rho^{(\mathrm{f})}(y,x) - \Big[ I^{(\mathrm{f})}_{(\rho)}(x,y)\gamma^\nu \rho^{(\mathrm{f})}(y,x)\Big]^\dagger \Big\}\Bigg) \\
&= -\mathrm{i}\, e^2 \Bigg\{ \qquad \mathrm{tr}\Big( I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)\gamma^\nu F^{(\mathrm{f})}(y,x)\Big) - \mathrm{tr}\Big( I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)\gamma^\nu F^{(\mathrm{f})}(y,x)\Big)^* \\
&\qquad + \frac{1}{4}\Big[ \mathrm{tr}\Big( I^{(\mathrm{f})}_{(\rho)}(x,y)\gamma^\nu \rho^{(\mathrm{f})}(y,x)\Big) - \mathrm{tr}\Big( I^{(\mathrm{f})}_{(\rho)}(x,y)\gamma^\nu \rho^{(\mathrm{f})}(y,x)\Big)^* \Big]\Bigg\} \\
&= 2e^2 \,\mathrm{Im}\Bigg( \mathrm{tr}\Big( I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)\gamma^\nu F^{(\mathrm{f})}(y,x)\Big) + \frac{1}{4}\,\mathrm{tr}\Big( I^{(\mathrm{f})}_{(\rho)}(x,y)\gamma^\nu \rho^{(\mathrm{f})}(y,x)\Big)\Bigg) \\
&= 2e^2 \,\mathrm{Im}\,\mathrm{tr}\Bigg( I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)\gamma^\nu F^{(\mathrm{f})}(y,x) + \frac{1}{4}\Big[ I^{(\mathrm{f})}_{(\rho)}(x,y)\gamma^\nu \rho^{(\mathrm{f})}(y,x)\Big]\Bigg)\,.
\end{aligned} \tag{5.13}
$$

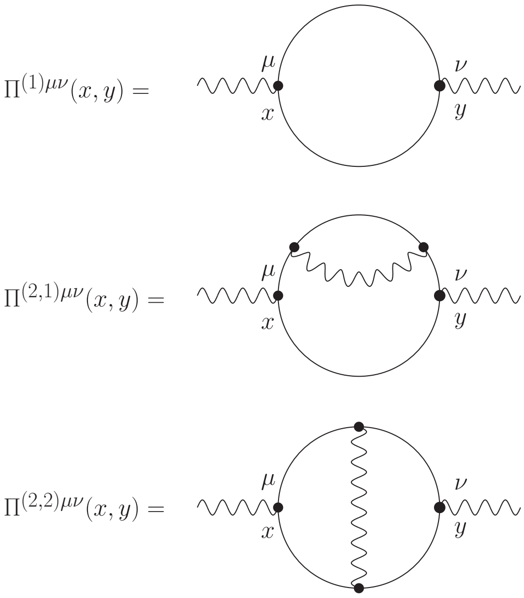



Figure 5.2: The one- and two-loop contributions $\Pi^{(1)\mu\nu}(x,y)$ and $\Pi^{(2,i)\mu\nu}(x,y)$ $(i = 1,2)$ to the perturbative photon self-energy. $\Pi^{(2,2)\mu\nu}(x,y)$ does not appear in the perturbative expansion of the two-loop 2PI effective action.

Similarly, one finds

$$\partial_{x\mu}\Pi^{\mu\nu}_{(\rho)}(x,y) = 2e^2\,\mathrm{Im}\,\mathrm{tr}\Big(I^{(\mathrm{f})}_{(\mathrm{F})}(x,y)\gamma^\nu\rho^{(\mathrm{f})}(y,x) - I^{(\mathrm{f})}_{(\rho)}(x,y)\gamma^\nu F^{(\mathrm{f})}(y,x)\Big)\,. \tag{5.14}$$

Therefore, the statistical and spectral photon self-energies are not transverse for a two-loop truncated 2PI effective action.[6]

Note, however, that *perturbatively*, the photon self-energy *is* in fact transverse:[7] Upon a perturbative expansion of the left-hand side, the lowest-order contribution to the memory integrals is $\mathcal{O}(e^2)$, so that the left-hand side is altogether $\mathcal{O}(e^4)$.

There is a more direct way to see the nontransversality of the photon self-energy in a two-loop truncation of the 2PI effective action by resorting to its diagrammatic expansion shown in Fig. (5.1). The corresponding self-energies are shown in Fig. (5.2). Since we

[6]The one-loop photon self-energy can be written as a current-current correlator, $\Pi^{\mu\nu}(x,y) = \langle J^\mu(x)J^\nu(y)\rangle$. The expectation value of the current itself is given by

$$\langle J^\mu(x)\rangle = 4e\int_{\boldsymbol{p}} F^{(\mathrm{f})\mu}_{\mathrm{V}}(x^0,x^0;\boldsymbol{p})\,.$$

Due to isotropy, it immediately follows that the spatial part vanishes identically. However, the temporal part vanishes as well by the Gauss law $\langle J^0(x)\rangle = \partial_i\langle F^{i0}(x)\rangle = 0$. It therefore follows that although $\partial_\mu\langle J^\mu(x)\rangle = 0$, one has $\partial_{x\mu}\langle J^\mu(x)J^\nu(y)\rangle \neq 0$.

[7]We have shown this earlier in this section for all Ward identities relating perturbative correlation functions.

know that the two-loop *perturbative* photon self-energy is transverse, we have

$$\partial_{x\mu}\Big[\Pi^{(2,1)\mu\nu}(x,y)+\Pi^{(2,2)\mu\nu}(x,y)\Big]=0\,. \tag{5.15}$$

Since $\Pi^{(2,2)\mu\nu}(x,y)$ is missing in the perturbative three-loop expansion of the two-loop 2PI effective action, to this order, it is given by:

$$\begin{aligned}
&\partial_{x\mu}\Pi^{(2,1)\mu\nu}(x,y)\\
&\quad=-\partial_{x\mu}\Pi^{(2,2)\mu\nu}(x,y)+\mathcal{O}(e^6)\\
&\quad=-e^4\int_{z,w}\partial_{x\mu}\,\mathrm{tr}\Big(\gamma^\mu S_0(x,z)\gamma^\rho S_0(z,y)\gamma^\nu S_0(y,w)\gamma^\sigma S_0(w,x)\Big)D_{0\rho\sigma}(z,w)+\mathcal{O}(e^6)\,.
\end{aligned} \tag{5.16}$$

Applying the free fermion EOMs, one has

$$\begin{aligned}
&\partial_{x\mu}\,\mathrm{tr}\Big(\gamma^\mu F_0^{(\mathrm{f})}(x,z)\gamma^\rho F_0^{(\mathrm{f})}(z,y)\gamma^\nu F_0^{(\mathrm{f})}(y,w)\gamma^\sigma F_0^{(\mathrm{f})}(w,x)\Big)\\
&\quad=-\,\mathrm{i}\,m^{(\mathrm{f})}\,\mathrm{tr}\Big(\gamma^\mu F_0^{(\mathrm{f})}(x,z)\gamma^\rho F_0^{(\mathrm{f})}(z,y)\gamma^\nu F_0^{(\mathrm{f})}(y,w)\gamma^\sigma F_0^{(\mathrm{f})}(w,x)\Big)\\
&\qquad+\mathrm{i}\,m^{(\mathrm{f})}\,\mathrm{tr}\Big(\gamma^\mu F_0^{(\mathrm{f})}(x,z)\gamma^\rho F_0^{(\mathrm{f})}(z,y)\gamma^\nu F_0^{(\mathrm{f})}(y,w)\gamma^\sigma F_0^{(\mathrm{f})}(w,x)\Big)\\
&\quad=0\,,
\end{aligned}$$

and similarly for the spectral part. It follows that

$$\partial_{x\mu}\Pi^{(2,1)\mu\nu}(x,y)=\mathcal{O}(e^6)\,. \tag{5.17}$$

The nontransversality of the photon self-energy for any finite truncation of the 2PI effective action is the reason that the auxiliary field correlation functions introduced previously are not free for any finite truncation.

# Chapter 6

# Numerical Implementation of the 2PI Equations of Motion

Due to the large number of components which have to be evolved and the corresponding memory resources and the very weak coupling of QED[1], a numerical solution of the 2PI EOMs up to times at which interesting physics happens is demanding.

In the following section, we will introduce the truncation of the 2PI effective action which we have used in our efforts to solve the EOMs numerically.

## 6.1 Two-Loop Truncation of the 2PI Effective Action

We will consider a two-loop truncation of the 2PI effective action. Even for such a simple truncation, the resulting self-energies turn out to be rather complicated structurally due to the large number of components it consists of.

### 6.1.1 Non-Contribution of the Nakanishi–Lautrup Field

Let us start with a short digression and show explicitly for our truncation that our previously made claim that the NL field does not contribute to the 2PI part of the 2PI effective action is in fact true. We will first show that the NL field does not contribute to the 2PI part of the 2PI effective action. Since there is no proper vertex for the NL field (and in particular no coupling to the fermions), correlation functions involving at least one NL field can only be attached to other correlation functions involving at least one NL field. There is then only one class of two-loop diagrams which could possibly contribute to the

[1] Of course, numerically one is not restricted to studying "physical QED", i. e. QED with a coupling constant whose value corresponds to the physical one, and one is free to chose any value. However, our use of a loop expansion of the 2PI effective action (see the next section) prohibits using larger values of the coupling constant.

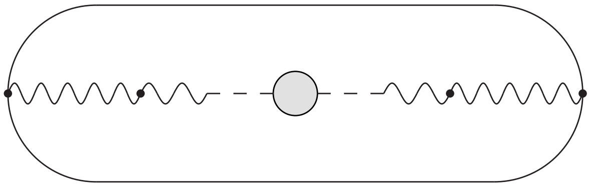


Figure 6.1: The only class of two-loop diagrams containing correlation functions involving the NL field. The blob contains only correlation functions involving at least one NL field (examples are given in the main text).

2PI part of the 2PI effective action, and it can be parametrized as (see Fig. 6.1)

$$\begin{aligned}&\mathcal{F}[D, D^{(AB)}, D^{(BA)}, D^{(BB)}, S]\\&\quad= -\frac{\mathrm{i}}{2}\int_{x,y,z,w} \Pi^{\mu\nu}(S;x,y)D_{\nu\rho}(x,z)\mathcal{N}^{\rho\sigma}(D, D^{(AB)}, D^{(BA)}, D^{(BB)}; z, w)D_{\sigma\nu}(w,y)\end{aligned}$$

with the fermion loop[2]

$$\Pi^{\mu\nu}(S;x,y) = e^2 \operatorname{tr}(\gamma^\mu S(x,y)\gamma^\nu S(y,x)) \tag{6.1}$$

and

$$\begin{aligned}&\mathcal{N}^{\rho\sigma}(D, D^{(AB)}, D^{(BA)}, D^{(BB)}; z, w)\\&\quad= \int_{u,v} D^{(AB)\rho}(z,u)\mathcal{M}(D, D^{(AB)}, D^{(BA)}, D^{(BB)}; u, v)D^{(BA)\sigma}(v,w)\end{aligned}$$

where $\mathcal{M}$ is an in general nonlocal scalar function depending potentially on all correlation functions involving at least one photon field. Examples are:

- $\mathcal{M}$ is local, i. e. depends on no correlation function at all:

$$\mathcal{M}(D, D^{(AB)}, D^{(BA)}, D^{(BB)}; u, v) = \delta^4(u-v)\,.$$

- $\mathcal{M}$ depends only on $D^{(BB)}$:

$$\mathcal{M}(D, D^{(AB)}, D^{(BA)}, D^{(BB)}; u, v) = D^{(BB)}(u,v)\,.$$

- $\mathcal{M}$ depends on the two mixed correlation functions:

$$\mathcal{M}(D, D^{(AB)}, D^{(BA)}, D^{(BB)}; u, v) = \int_s D^{(BA)}_\lambda(u,s)D^{(AB)\lambda}(s,v)\,.$$

In general, however, $\mathcal{M}$ can depend on an arbitrary number of correlation functions.

The important point now is to notice that if we would like to include at least one correlator involving an NL field, we need to attach *two* photon propagators coupling to

[2]We will shortly see that this is exactly the photon self-energy. Here, however, it is just used as an abbreviation.

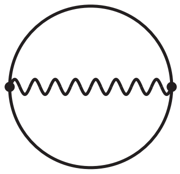


Figure 6.2: The only diagram contributing to the 2PI part of the effective action at two-loop level (6.2). Note that the lines correspond to full propagators, but the vertices are the classical ones.

the fermion propagators. But then, as can easily be seen in Fig. 6.1, the diagram is not 2PI any more, since cutting the two photon propagators yields a disconnected diagram consisting of the fermion loop and the string of correlation functions involving at least one NL field. Explicitly:

$$\frac{\delta^2\mathcal{F}[S,D,D^{(AB)},D^{(BA)},D^{(BB)}]}{\delta D_{\mu\nu}(x,y)\delta D_{\rho\sigma}(z,w)} = -\frac{\mathrm{i}}{2}\Big[\ \Pi^{\mu\sigma}(S;x,w)\,\mathcal{N}^{\nu\rho}(D^{(AB)},D^{(BA)},D^{(BB)};y,z) + \Pi^{\rho\nu}(S;z,y)\,\mathcal{N}^{\sigma\mu}(D^{(AB)},D^{(BA)},D^{(BB)};w,x)\Big]\,,$$

which is clearly disconnected. Therefore, any diagram which depends on a correlation function which involves at least one NL field cannot contribute to the 2PI part of the 2PI effective action and hence only appears in the free gas part.

The 2PI part of the 2PI effective action is then (for vanishing field expectation values) given by:

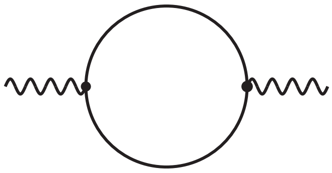
$$\begin{aligned}\Gamma_2[S,D] &= -\frac{\mathrm{i}\,e^2}{2}\int_{x,y}\mathcal{F}^{\mu\nu}(S;x,y)D_{\mu\nu}(x,y)\\ &= -\frac{\mathrm{i}\,e^2}{2}\int_{x,y}\mathrm{tr}(\gamma^\mu S(x,y)\gamma^\nu S(y,x))D_{\mu\nu}(x,y)\,,\end{aligned} \tag{6.2}$$

which corresponds to the single diagram shown in Fig. (6.2) in a diagrammatic representation. Combining the photon field and the NL field into a composite field $(\widetilde{A}_m) = (A_\mu, B)$ (with $m = 0,\ldots,4$ and $\mu = 0,\ldots,3$, so that $\widetilde{A}_\mu = A_\mu$, $\widetilde{A}_4 = B$), the complete 2PI effective action then reads:

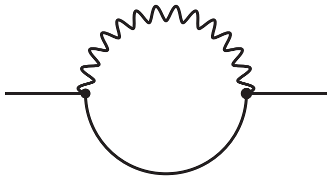
$$\Gamma_{2\mathrm{PI}}[\widetilde{D},S] = \frac{\mathrm{i}}{2}\,\mathrm{Tr}\ln\widetilde{D}^{-1} + \frac{\mathrm{i}}{2}\,\mathrm{Tr}\Big(\widetilde{D}_0^{-1}\widetilde{D}\Big) - \mathrm{i}\,\mathrm{Tr}\ln S^{-1} - \mathrm{i}\,\mathrm{Tr}\Big(S_0^{-1}S\Big) + \Gamma_2[S,D] \tag{6.3}$$

According to (3.8) and (3.9), the self-energies are then given by:

$$\Pi^{\mu\nu}(x,y) = e^2\,\mathrm{tr}(\gamma^\mu S(x,y)\gamma^\nu S(y,x))\,, \tag{6.4}$$
$$\Sigma(x,y) = -e^2\gamma^\mu S(x,y)\gamma^\nu D_{\mu\nu}(x,y)\,, \tag{6.5}$$



(a) photon self-energy



(b) fermion self-energy

Figure 6.3: The one-loop self-energies (6.4) and (6.5).

for the photon and for the fermion, respectively.[3] Their diagrammatic representation is shown in Fig. (6.3).

In addition, we have

$$\widetilde{\Pi}^{\mu 4}(x,y) = \widetilde{\Pi}^{4\mu}(x,y) = \widetilde{\Pi}^{44}(x,y) = 0\,. \tag{6.7}$$

In the following subsections, we will decompose the self-energies into components according to the given symmetries.

### 6.1.2 Photon Self-Energy

Evaluating the traces, the photon self-energy (6.4) reads in the basis introduced above:

$$\begin{aligned}
&\frac{1}{4e^2}\Pi^{\mu\nu}(x,y)\\
&= \quad \delta_0^\mu\delta_0^\nu\Big[S_{\mathrm{S}}(x,y)S_{\mathrm{S}}(y,x) - \widetilde{S}_{\mathrm{V}}^0(x,y)\widetilde{S}_{\mathrm{V}}^0(y,x) - S_{\mathrm{V}i}(x,y)S_{\mathrm{V}}^i(y,x) - S_{\mathrm{T}i}(x,y)S_{\mathrm{T}}^{i0}(y,x)\Big]\\
&\quad + \mathrm{i}\,\delta_0^\mu\delta_i^\nu\Big[\widetilde{S}_{\mathrm{V}}^0(x,y)S_{\mathrm{V}}^i(y,x) + S_{\mathrm{V}}^i(x,y)\widetilde{S}_{\mathrm{V}}^0(y,x) + S_{\mathrm{S}}(x,y)S_{\mathrm{T}}^{i0}(y,x) - S_{\mathrm{T}}^{i0}(x,y)S_{\mathrm{S}}(y,x)\Big]\\
&\quad + \mathrm{i}\,\delta_i^\mu\delta_0^\nu\Big\{\widetilde{S}_{\mathrm{V}}^0(x,y)S_{\mathrm{V}}^i(y,x) + S_{\mathrm{V}}^i(x,y)\widetilde{S}_{\mathrm{V}}^0(y,x) - \Big[S_{\mathrm{S}}(x,y)S_{\mathrm{T}}^{i0}(y,x) - S_{\mathrm{T}}^{i0}(x,y)S_{\mathrm{S}}(y,x)\Big]\Big\}\\
&\quad + \delta_i^\mu\delta_j^\nu\Big\{S_{\mathrm{V}}^i(x,y)S_{\mathrm{V}}^j(y,x) + S_{\mathrm{V}}^j(x,y)S_{\mathrm{V}}^i(y,x) - \Big[S_{\mathrm{T}}^{i0}(x,y)S_{\mathrm{T}}^{j0}(y,x) + S_{\mathrm{T}}^{j0}(x,y)S_{\mathrm{T}}^{i0}(y,x)\Big]\\
&\qquad\quad + g^{ij}\Big[S_{\mathrm{S}}(x,y)S_{\mathrm{S}}(y,x) + \widetilde{S}_{\mathrm{V}}^0(x,y)\widetilde{S}_{\mathrm{V}}^0(y,x)\\
&\qquad\qquad - S_{\mathrm{V}k}(x,y)S_{\mathrm{V}}^k(y,x) + S_{\mathrm{T}k0}(x,y)S_{\mathrm{T}}^{k0}(y,x)\Big]\Big\}\,.
\end{aligned} \tag{6.8}$$

[3]Note that this implies that the 2PI-part of the 2PI effective action can also be written as

$$\Gamma_2[S,D] = -\frac{\mathrm{i}}{2}\int_{x,y}\Pi^{\mu\nu}(x,y)D_{\mu\nu}(x,y) = \frac{\mathrm{i}}{2}\int_{x,y}\mathrm{tr}(\Sigma(x,y)S(y,x))\,. \tag{6.6}$$

After a partial Fourier transformation with respect to space, the isotropic components of the statistical and spectral photon self-energies read:

$$
\begin{aligned}
&\frac{1}{4e^2}\,\Pi_{(F)\mathrm{S}}(t,t';|\boldsymbol{p}|)\\
&\quad=\int_{\boldsymbol{q}}\Bigg\{\; F^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\\
&\qquad\qquad-\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\Big[\; F^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+F^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\Bigg\}\\
&\qquad-\frac{1}{4}\int_{\boldsymbol{q}}\Bigg\{\; \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\\
&\qquad\qquad-\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\Big[\; \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\Bigg\}\,,
\end{aligned}
\tag{6.9a}
$$

$$
\begin{aligned}
&\frac{1}{8e^2}\,\widetilde{\Pi}_{(F)\mathrm{V}_1}(t,t';|\boldsymbol{p}|)\\
&\quad=\int_{\boldsymbol{q}}\Bigg\{-\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,F^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)+\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,F^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{T}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\Bigg\}\\
&\qquad-\frac{1}{4}\int_{\boldsymbol{q}}\Bigg\{-\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)+\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{f})}_{\mathrm{T}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\Bigg\}\,,
\end{aligned}
\tag{6.9b}
$$

$$
\begin{aligned}
&\frac{1}{8e^2}\,\widetilde{\Pi}_{(F)\mathrm{V}_2}(t,t';|\boldsymbol{p}|)\\
&\quad=\int_{\boldsymbol{q}}\Bigg\{-\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,F^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)-\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,F^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{T}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\Bigg\}\\
&\qquad-\frac{1}{4}\int_{\boldsymbol{q}}\Bigg\{-\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)-\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{f})}_{\mathrm{T}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\Bigg\}\,,
\end{aligned}
\tag{6.9c}
$$

$$
\begin{aligned}
&\frac{1}{4e^2}\,\Pi_{(F)\mathrm{L}}(t,t';|\boldsymbol{p}|)\\
&\quad=\int_{\boldsymbol{q}}\Bigg\{\; F^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{S}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)-\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\\
&\qquad\qquad-\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\Big[\; F^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)-F^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{T}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\
&\qquad\qquad+2\,\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\Big[\; F^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad - F^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{T}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\Big]\Bigg\} \\
&-\frac{1}{4}\int_{\boldsymbol{q}}\Bigg\{\ \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{f})}_{\mathrm{S}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|) - \widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|) \\
&\qquad - \frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\Big[\ \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{f})}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|) - \rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{f})}_{\mathrm{T}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\Big] \\
&\qquad + 2\,\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\Big[\ \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{f})}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|) \\
&\qquad\qquad - \rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{f})}_{\mathrm{T}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\Big]\Bigg\}\,, 
\end{aligned}
\tag{6.9d}
$$

$$
\begin{aligned}
&\frac{1}{4e^2}\,\Pi_{(F)\mathrm{T}}(t,t';|\boldsymbol{p}|) \\
&= \int_{\boldsymbol{q}}\Bigg\{\ F^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{S}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|) - \widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|) \\
&\qquad - \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\Big[\ F^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|) \\
&\qquad\qquad - F^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{T}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\Big]\Bigg\} \\
&-\frac{1}{4}\int_{\boldsymbol{q}}\Bigg\{\ \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{f})}_{\mathrm{S}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|) - \widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|) \\
&\qquad - \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\Big[\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{f})}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|) \\
&\qquad\qquad - \rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{f})}_{\mathrm{T}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\Big]\Bigg\}\,, 
\end{aligned}
\tag{6.9e}
$$

$$
\begin{aligned}
&\frac{1}{8e^2}\,\Pi_{(\rho)\mathrm{S}}(t,t';|\boldsymbol{p}|) \\
&= \int_{\boldsymbol{q}}\Bigg\{\ \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|) + \widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|) \\
&\qquad - \frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\Big[\ \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|) + \rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\Bigg\}\,, 
\end{aligned}
\tag{6.9f}
$$

$$
\begin{aligned}
&\frac{1}{8e^2}\,\widetilde{\Pi}_{(\rho)\mathrm{V}_1}(t,t';|\boldsymbol{p}|) \\
&= \int_{\boldsymbol{q}}\Bigg\{-\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\Big[\ \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|) + F^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\Big] \\
&\qquad + \frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\Big[\ \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{T}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|) + F^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{f})}_{\mathrm{T}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\Big]\Bigg\}\,,
\end{aligned}
$$

$$\tag{6.9g}$$

$$
\begin{aligned}
&\frac{1}{8e^2}\,\widetilde{\Pi}_{(\rho)\mathrm{V}_2}(t,t';|\boldsymbol{p}|)\\
&\quad=\int_{\boldsymbol{q}}\Bigg\{-\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\Big[\ \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)+F^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\
&\qquad\qquad-\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\Big[\ \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{T}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)+F^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{f})}_{\mathrm{T}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\Big]\Bigg\}\,,
\end{aligned}\tag{6.9h}
$$

$$
\begin{aligned}
&\frac{1}{8e^2}\,\Pi_{(\rho)\mathrm{L}}(t,t';|\boldsymbol{p}|)\\
&\quad=\int_{\boldsymbol{q}}\Bigg\{\ \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{S}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)-\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\\
&\qquad\qquad-\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\Big[\ \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)-\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{T}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\
&\qquad\qquad+2\,\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\Big[\ \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\\
&\qquad\qquad\qquad\qquad-\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{T}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\Big]\Bigg\}\,,
\end{aligned}\tag{6.9i}
$$

$$
\begin{aligned}
&\frac{1}{8e^2}\,\Pi_{(\rho)\mathrm{T}}(t,t';|\boldsymbol{p}|)\\
&\quad=\int_{\boldsymbol{q}}\Bigg\{\ \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{S}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)-\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\\
&\qquad\qquad-\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\Big[\ \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{V}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\\
&\qquad\qquad\qquad\qquad-\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{f})}_{\mathrm{T}}(t',t;|\boldsymbol{p}-\boldsymbol{q}|)\Big]\Bigg\}\,.
\end{aligned}\tag{6.9j}
$$

Employing our definition of generalized convolutions (see App. C), the self-energies can be written in a much more compact way:

For the statistical components, we obtain:

$$
\begin{aligned}
\frac{1}{4e^2}\,\Pi_{(F)\mathrm{S}}(t,t';\cdot)=\ &F^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot)*_{\mathrm{SS}}F^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot)+\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)*_{\mathrm{SS}}\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)\\
&-F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot)*_{\mathrm{VV}_1}F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot)-F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)*_{\mathrm{VV}_1}F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)\\
&-\frac{1}{4}\Big[\ \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot)*_{\mathrm{SS}}\rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot)+\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)*_{\mathrm{SS}}\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)\\
&\qquad-\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot)*_{\mathrm{VV}_1}\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot)-\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)*_{\mathrm{VV}_1}\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)\Big]\,,
\end{aligned}\tag{6.10a}
$$

$$
\begin{aligned}
\frac{1}{8e^2}\,\widetilde{\Pi}_{(F)\mathrm{V}_1}(t,t';\cdot)=&-F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot)*_{\mathrm{VS}}\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)+F^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot)*_{\mathrm{SV}}F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)\\
&-\frac{1}{4}\Big[-\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot)*_{\mathrm{VS}}\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)+\rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot)*_{\mathrm{SV}}\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)\Big]\,,
\end{aligned}\tag{6.10b}
$$

$$\begin{aligned}\frac{1}{8e^2}\,\widetilde{\Pi}_{(F)\mathrm{V}_2}(t,t';\cdot) = & -F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VS}} \widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot) - F^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SV}} F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)\\ & - \frac{1}{4}\Big[-\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VS}} \widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot) - \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SV}} \rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)\Big], \end{aligned} \tag{6.10c}$$

$$\begin{aligned}\frac{1}{4e^2}\,\Pi_{(F)\mathrm{L}}(t,t';\cdot) = & \quad F^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SS}} F^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) - \widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{SS}} \widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)\\ & - F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VV}_1} F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) + F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VV}_1} F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)\\ & + 2F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VV}_2} F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) - 2F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VV}_2} F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)\\ & - \frac{1}{4}\Big[\quad \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SS}} \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) - \widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{SS}} \widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)\\ & \qquad - \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VV}_1} \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) + \rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VV}_1} \rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)\\ & \qquad + 2\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VV}_2} \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) - 2\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VV}_2} \rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)\Big], \end{aligned} \tag{6.10d}$$

$$\begin{aligned}\frac{1}{4e^2}\,\Pi_{(F)\mathrm{T}}(t,t';\cdot) = & \quad F^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SS}} F^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) - \widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{SS}} \widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)\\ & - F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VV}_2} F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) + F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VV}_2} F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)\\ & - \frac{1}{4}\Big[\quad \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SS}} \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) - \widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{SS}} \widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)\\ & \qquad - \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VV}_2} \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) + \rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VV}_2} \rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)\Big]. \end{aligned} \tag{6.10e}$$

Note that

$$\frac{1}{8e^2}\,\widetilde{\Pi}_{(F)\mathrm{V}_1}(t,t;\cdot) = F^{(\mathrm{f})}_{\mathrm{S}}(t,t;\cdot) *_{\mathrm{SV}} F^{(\mathrm{f})}_{\mathrm{T}}(t,t;\cdot) = -\frac{1}{8e^2}\,\widetilde{\Pi}_{(F)\mathrm{V}_2}(t,t;\cdot)\,, \tag{6.11}$$

as it has to be.

For the spectral components, we obtain:

$$\begin{aligned}\frac{1}{8e^2}\,\Pi_{(\rho)\mathrm{S}}(t,t';\cdot) = & \quad \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SS}} F^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) + \widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{SS}} \widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)\\ & - \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VV}_1} F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) - \rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VV}_1} F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)\,, \end{aligned} \tag{6.12a}$$

$$\begin{aligned}\frac{1}{8e^2}\,\widetilde{\Pi}_{(\rho)\mathrm{V}_1}(t,t';\cdot) = & -\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VS}} \widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot) - F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VS}} \widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)\\ & + \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SV}} F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) + F^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SV}} \rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)\,, \end{aligned} \tag{6.12b}$$

$$\begin{aligned}\frac{1}{8e^2}\,\widetilde{\Pi}_{(\rho)\mathrm{V}_2}(t,t';\cdot) = & -\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VS}} \widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot) - F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VS}} \widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)\\ & - \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SV}} F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) - F^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SV}} \rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)\,, \end{aligned} \tag{6.12c}$$

$$\begin{aligned}\frac{1}{8e^2}\,\Pi_{(\rho)\mathrm{L}}(t,t';\cdot) = & \quad \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SS}} F^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) - \widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{SS}} \widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)\\ & - \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VV}_1} F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) + \rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VV}_1} F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)\\ & + 2\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VV}_2} F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) - 2\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VV}_2} F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)\,, \end{aligned} \tag{6.12d}$$

$$\frac{1}{8e^2}\,\Pi_{(\rho)\mathrm{T}}(t,t';\cdot) = \quad \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SS}} F^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) - \widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{SS}} \widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)$$

$$- \rho_{\mathrm{V}}^{(\mathrm{f})}(t,t';\cdot) *_{\mathrm{VV}_2} F_{\mathrm{V}}^{(\mathrm{f})}(t,t';\cdot) + \rho_{\mathrm{T}}^{(\mathrm{f})}(t,t';\cdot) *_{\mathrm{VV}_2} F_{\mathrm{T}}^{(\mathrm{f})}(t,t';\cdot)\,. \tag{6.12e}$$

Note that

$$\frac{1}{8e^2}\,\widetilde{\Pi}_{(\rho)\mathrm{V}_1}(t,t;\cdot) = -F_{\mathrm{V}}^{(\mathrm{f})}(t,t;\cdot) *_{\mathrm{VS}} 1 = \frac{1}{8e^2}\,\widetilde{\Pi}_{(\rho)\mathrm{V}_2}(t,t;\cdot)\,, \tag{6.13}$$

as it has to be. In fact,

$$\frac{1}{8e^2}\,\widetilde{\Pi}_{(\rho)\mathrm{V}_1}(t,t;|\boldsymbol{p}|) = -\int_{\boldsymbol{q}} \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,F_{\mathrm{V}}^{(\mathrm{f})}(t,t;|\boldsymbol{q}|) = -\frac{1}{4\pi^2}\int_0^\infty \mathrm{d}q\,q^2 F_{\mathrm{V}}^{(\mathrm{f})}(t,t;q)\int_{-1}^1 \mathrm{d}x\,x = 0\,. \tag{6.14}$$

### 6.1.3 Fermion Self-Energy

Similarly, evaluating the fermion self-energy (6.5) yields

$$\frac{1}{e^2}\Sigma(x,y) = -\Big[\gamma^\mu\gamma^\nu S_{\mathrm{S}}(x,y) + \gamma^\mu\gamma^\rho\gamma^\nu S_{\mathrm{V}\rho}(x,y) + \mathrm{i}\,\gamma^\mu\gamma^i\gamma^0\gamma^\nu S_{\mathrm{T}i0}(x,y)\Big] D_{\mu\nu}(x,y)\,, \tag{6.15}$$

which can be decomposed according to:

$$\frac{1}{e^2}\Sigma_{\mathrm{S}}(x,y) = -S_{\mathrm{S}}(x,y)\Big[D_{\mathrm{S}}(x,y) + g^{ij}D_{ij}(x,y)\Big] + S_{\mathrm{T}i0}(x,y)\Big[\widetilde{D}^{i0}(x,y) - \widetilde{D}^{0i}(x,y)\Big]\,, \tag{6.16}$$

$$\frac{1}{e^2}\widetilde{\Sigma}_{\mathrm{V}}^0(x,y) = -\widetilde{S}_{\mathrm{V}}^0(x,y)\Big[D_{\mathrm{S}}(x,y) - g^{ij}D_{ij}(x,y)\Big] - S_{\mathrm{V}}^i(x,y)\Big[\widetilde{D}_{0i}(x,y) + \widetilde{D}_{i0}(x,y)\Big]\,, \tag{6.17}$$

$$\begin{aligned}\frac{1}{e^2}\Sigma_{\mathrm{V}}^i(x,y) = {}& S_{\mathrm{V}i}(x,y)\Big[D_{\mathrm{S}}(x,y) + g^{jk}D_{jk}(x,y)\Big] - 2S_{\mathrm{V}}^j(x,y)D_{ij}(x,y)\\ &+ \widetilde{S}_{\mathrm{V}}^0(x,y)\Big[\widetilde{D}_{i0}(x,y) + \widetilde{D}_{0i}(x,y)\Big]\,,\end{aligned} \tag{6.18}$$

$$\begin{aligned}\frac{1}{e^2}\Sigma_{\mathrm{T}}^{i0}(x,y) = {}& \quad S_{\mathrm{T}i0}(x,y)\Big[D_{\mathrm{S}}(x,y) - g^{jk}D_{jk}(x,y)\Big] + 2S_{\mathrm{T}}^{j0}(x,y)D_{ij}(x,y)\\ &- S_{\mathrm{S}}(x,y)\Big[\widetilde{D}_{i0}(x,y) - \widetilde{D}_{0i}(x,y)\Big]\,.\end{aligned} \tag{6.19}$$

After a partial Fourier transformation with respect to space, the isotropic components of the statistical and spectral fermion self-energies read:

$$\begin{aligned}&\frac{1}{e^2}\,\Sigma_{(F)\mathrm{S}}(t,t';|\boldsymbol{p}|)\\ &\quad = \int_{\boldsymbol{q}}\Big\{-F_{\mathrm{S}}^{(\mathrm{f})}(t,t';|\boldsymbol{q}|)\Big[F_{\mathrm{S}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|) + 2F_{\mathrm{T}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|) + F_{\mathrm{L}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\ &\qquad\qquad + \frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,F_{\mathrm{T}}^{(\mathrm{f})}(t,t';|\boldsymbol{q}|)\Big[-\widetilde{F}_{\mathrm{V}_1}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|) + \widetilde{F}_{\mathrm{V}_2}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\Big\}\\ &\qquad -\frac{1}{4}\int_{\boldsymbol{q}}\Big\{-\rho_{\mathrm{S}}^{(\mathrm{f})}(t,t';|\boldsymbol{q}|)\Big[\rho_{\mathrm{S}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|) + 2\rho_{\mathrm{T}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|) + \rho_{\mathrm{L}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\ &\qquad\qquad + \frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,\rho_{\mathrm{T}}^{(\mathrm{f})}(t,t';|\boldsymbol{q}|)\Big[-\widetilde{\rho}_{\mathrm{V}_1}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|) + \widetilde{\rho}_{\mathrm{V}_2}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\Big\}\,,\end{aligned} \tag{6.20a}$$

$$
\begin{aligned}
&\frac{1}{e^2}\,\widetilde{\Sigma}^0_{(F)\mathrm{V}}(t,t';|\boldsymbol{p}|)\\
&\quad=\int_{\boldsymbol{q}}\bigg\{\;\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\Big[-F^{(\mathrm{g})}_{\mathrm{S}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+2F^{(\mathrm{g})}_{\mathrm{T}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+F^{(\mathrm{g})}_{\mathrm{L}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\
&\qquad\quad+\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,F^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\Big[\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\bigg\}\\
&\qquad-\frac{1}{4}\int_{\boldsymbol{q}}\bigg\{\;\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\Big[-\rho^{(\mathrm{g})}_{\mathrm{S}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+2\rho^{(\mathrm{g})}_{\mathrm{T}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\
&\qquad\quad+\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\Big[\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\bigg\}\,, \qquad (6.20\mathrm{b})
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{e^2}\,\Sigma_{(F)\mathrm{V}}(t,t';|\boldsymbol{p}|)\\
&\quad=\int_{\boldsymbol{q}}\bigg\{\;\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,F^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\Big[F^{(\mathrm{g})}_{\mathrm{S}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+2F^{(\mathrm{g})}_{\mathrm{T}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+F^{(\mathrm{g})}_{\mathrm{L}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\
&\qquad\quad-2\,\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,F^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{g})}_{\mathrm{L}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\\
&\qquad\quad+\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\Big[\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\bigg\}\\
&\qquad-\frac{1}{4}\int_{\boldsymbol{q}}\bigg\{\;\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\Big[\rho^{(\mathrm{g})}_{\mathrm{S}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+2\rho^{(\mathrm{g})}_{\mathrm{T}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\
&\qquad\quad-2\,\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\\
&\qquad\quad+\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}t,t';|\boldsymbol{q}|)\Big[\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\bigg\}\,, \qquad (6.20\mathrm{c})
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{e^2}\,\Sigma_{(F)\mathrm{T}}(t,t';|\boldsymbol{p}|)\\
&\quad=\int_{\boldsymbol{q}}\bigg\{\;\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,F^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)\Big[F^{(\mathrm{g})}_{\mathrm{S}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)-2F^{(\mathrm{g})}_{\mathrm{T}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)-F^{(\mathrm{g})}_{\mathrm{L}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\
&\qquad\quad+2\,\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,F^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{g})}_{\mathrm{L}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\\
&\qquad\quad+\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,F^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)\Big[-\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\bigg\}\\
&\qquad-\frac{1}{4}\int_{\boldsymbol{q}}\bigg\{\;\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)\Big[\rho^{(\mathrm{g})}_{\mathrm{S}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)-2\rho^{(\mathrm{g})}_{\mathrm{T}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)-\rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\
&\qquad\quad+2\,\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\\
&\qquad\quad+\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)\Big[-\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\bigg\}\,, \qquad (6.20\mathrm{d})
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{e^2}\,\Sigma_{(\rho)\mathrm{S}}(t,t';|\boldsymbol{p}|)\\
&\quad=\int_{\boldsymbol{q}}\bigg\{-F_{\mathrm{S}}^{(\mathrm{f})}(t,t';|\boldsymbol{q}|)\Big[\rho_{\mathrm{S}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+2\rho_{\mathrm{T}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\rho_{\mathrm{L}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\
&\qquad\quad+\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,F_{\mathrm{T}}^{(\mathrm{f})}(t,t';|\boldsymbol{q}|)\Big[-\widetilde{\rho}_{\mathrm{V}_1}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\widetilde{\rho}_{\mathrm{V}_2}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\bigg\}\\
&\qquad\quad-\rho_{\mathrm{S}}^{(\mathrm{f})}(t,t';|\boldsymbol{q}|)\Big[F_{\mathrm{S}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+2F_{\mathrm{T}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+F_{\mathrm{L}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\
&\qquad\quad+\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,\rho_{\mathrm{T}}^{(\mathrm{f})}(t,t';|\boldsymbol{q}|)\Big[-\widetilde{F}_{\mathrm{V}_1}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\widetilde{F}_{\mathrm{V}_2}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\bigg\}\,,
\end{aligned}
\tag{6.20e}
$$

$$
\begin{aligned}
&\frac{1}{e^2}\,\widetilde{\Sigma}^0_{(\rho)\mathrm{V}}(t,t';|\boldsymbol{p}|)\\
&\quad=\int_{\boldsymbol{q}}\bigg\{\;\widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(t,t';|\boldsymbol{q}|)\Big[-\rho_{\mathrm{S}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+2\rho_{\mathrm{T}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\rho_{\mathrm{L}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\
&\qquad\quad+\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,F_{\mathrm{V}}^{(\mathrm{f})}(t,t';|\boldsymbol{q}|)\Big[\widetilde{\rho}_{\mathrm{V}_1}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\widetilde{\rho}_{\mathrm{V}_2}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\bigg\}\\
&\qquad\quad+\widetilde{\rho}_{\mathrm{V}}^{(\mathrm{f})0}(t,t';|\boldsymbol{q}|)\Big[-F_{\mathrm{S}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+2F_{\mathrm{T}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+F_{\mathrm{L}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\
&\qquad\quad+\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,\rho_{\mathrm{V}}^{(\mathrm{f})}(t,t';|\boldsymbol{q}|)\Big[\widetilde{F}_{\mathrm{V}_1}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\widetilde{F}_{\mathrm{V}_2}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\bigg\}\,,
\end{aligned}
\tag{6.20f}
$$

$$
\begin{aligned}
&\frac{1}{e^2}\,\Sigma_{(\rho)\mathrm{V}}(t,t';|\boldsymbol{p}|)\\
&\quad=\int_{\boldsymbol{q}}\bigg\{\;\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,F_{\mathrm{V}}^{(\mathrm{f})}(t,t';|\boldsymbol{q}|)\Big[\rho_{\mathrm{S}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+2\rho_{\mathrm{T}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\rho_{\mathrm{L}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\
&\qquad\quad-2\,\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,F_{\mathrm{V}}^{(\mathrm{f})}(t,t';|\boldsymbol{q}|)\rho_{\mathrm{L}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\\
&\qquad\quad+\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(t,t';|\boldsymbol{q}|)\Big[\widetilde{\rho}_{\mathrm{V}_1}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\widetilde{\rho}_{\mathrm{V}_2}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\bigg\}\\
&\qquad\quad+\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,\rho_{\mathrm{V}}^{(\mathrm{f})}(t,t';|\boldsymbol{q}|)\Big[F_{\mathrm{S}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+2F_{\mathrm{T}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+F_{\mathrm{L}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\
&\qquad\quad-2\,\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,\rho_{\mathrm{V}}^{(\mathrm{f})}(t,t';|\boldsymbol{q}|)F_{\mathrm{L}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\\
&\qquad\quad+\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\widetilde{\rho}_{\mathrm{V}}^{(\mathrm{f})0}(t,t';|\boldsymbol{q}|)\Big[\widetilde{F}_{\mathrm{V}_1}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\widetilde{F}_{\mathrm{V}_2}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\bigg\}\,,
\end{aligned}
\tag{6.20g}
$$

$$
\begin{aligned}
&\frac{1}{e^2}\,\Sigma_{(\rho)\mathrm{T}}(t,t';|\boldsymbol{p}|)\\
&\quad=\int_{\boldsymbol{q}}\bigg\{\;\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,F_{\mathrm{T}}^{(\mathrm{f})}(t,t';|\boldsymbol{q}|)\Big[\rho_{\mathrm{S}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)-2\rho_{\mathrm{T}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)-\rho_{\mathrm{L}}^{(\mathrm{g})}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\,\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,F^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)\rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\\
&+ \frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,F^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)\Big[-\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\Big\}\\
&+ \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)\Big[F^{(\mathrm{g})}_{\mathrm{S}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)-2F^{(\mathrm{g})}_{\mathrm{T}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)-F^{(\mathrm{g})}_{\mathrm{L}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\\
&+ 2\,\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';|\boldsymbol{q}|)F^{(\mathrm{g})}_{\mathrm{L}}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\\
&+ \frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';|\boldsymbol{q}|)\Big[-\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)+\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';|\boldsymbol{p}-\boldsymbol{q}|)\Big]\Big\}\,. \qquad (6.20\mathrm{h})
\end{aligned}
$$

Again employing the notation of the generalized convolutions, we obtain for the statistical components:

$$
\begin{aligned}
\frac{1}{e^2}\,\Sigma_{(F)\mathrm{S}}(t,t';\cdot) = &-F^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot)*_{\mathrm{SS}}\Big[F^{(\mathrm{g})}_{\mathrm{S}}(t,t';\cdot)+2F^{(\mathrm{g})}_{\mathrm{T}}(t,t';\cdot)+F^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\Big]\\
&+F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)*_{\mathrm{VV}_1}\Big[-\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';\cdot)+\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';\cdot)\Big]\\
&-\frac{1}{4}\Big\{-\rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot)*_{\mathrm{SS}}\Big[\rho^{(\mathrm{g})}_{\mathrm{S}}(t,t';\cdot)+2\rho^{(\mathrm{g})}_{\mathrm{T}}(t,t';\cdot)+\rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\Big]\\
&\qquad+\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot)*_{\mathrm{VV}_1}\Big[-\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';\cdot)+\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';\cdot)\Big]\Big\}\,, \qquad (6.21\mathrm{a})\\
\frac{1}{e^2}\,\widetilde{\Sigma}^0_{(F)\mathrm{V}}(t,t';\cdot) = &\quad\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)*_{\mathrm{SS}}\Big[-F^{(\mathrm{g})}_{\mathrm{S}}(t,t';\cdot)+2F^{(\mathrm{g})}_{\mathrm{T}}(t,t';\cdot)+F^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\Big]\\
&+F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot)*_{\mathrm{VV}_1}\Big[\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';\cdot)+\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';\cdot)\Big]\\
&-\frac{1}{4}\Big\{\quad\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)*_{\mathrm{SS}}\Big[-\rho^{(\mathrm{g})}_{\mathrm{S}}(t,t';\cdot)+2\rho^{(\mathrm{g})}_{\mathrm{T}}(t,t';\cdot)+\rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\Big]\\
&\qquad+\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot)*_{\mathrm{VV}_1}\Big[\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';\cdot)+\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';\cdot)\Big]\Big\}\,, \qquad (6.21\mathrm{b})\\
\frac{1}{e^2}\,\Sigma_{(F)\mathrm{V}}(t,t';\cdot) = &\quad F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot)*_{\mathrm{VS}}\Big[F^{(\mathrm{g})}_{\mathrm{S}}(t,t';\cdot)+2F^{(\mathrm{g})}_{\mathrm{T}}(t,t';\cdot)+F^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\Big]\\
&-2F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot)*_{\mathrm{VT}}F^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\\
&+\widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)*_{\mathrm{SV}}\Big[\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';\cdot)+\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';\cdot)\Big]\\
&-\frac{1}{4}\Big\{\quad\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot)*_{\mathrm{VS}}\Big[\rho^{(\mathrm{g})}_{\mathrm{S}}(t,t';\cdot)+2\rho^{(\mathrm{g})}_{\mathrm{T}}(t,t';\cdot)+\rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\Big]\\
&\qquad-2\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot)*_{\mathrm{VT}}\rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\\
&\qquad+\widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot)*_{\mathrm{SV}}\Big[\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';\cdot)+\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';\cdot)\Big]\Big\}\,, \qquad (6.21\mathrm{c})
\end{aligned}
$$

$$\begin{aligned}
\frac{1}{e^2}\,\Sigma_{(F)\mathrm{T}}(t,t';\cdot) = {}& F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VS}} \Big[F^{(\mathrm{g})}_{\mathrm{S}}(t,t';\cdot) - 2F^{(\mathrm{g})}_{\mathrm{T}}(t,t';\cdot) - F^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\Big] \\
& + 2F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VT}} F^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot) \\
& + F^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SV}} \Big[-\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';\cdot) + \widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';\cdot)\Big] \\
& - \frac{1}{4}\Big\{ \rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VS}} \Big[\rho^{(\mathrm{g})}_{\mathrm{S}}(t,t';\cdot) - 2\rho^{(\mathrm{g})}_{\mathrm{T}}(t,t';\cdot) - \rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\Big] \\
& \qquad + 2\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VT}} \rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot) \\
& \qquad + \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SV}} \Big[-\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';\cdot) + \widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';\cdot)\Big]\Big\}\,.
\end{aligned} \tag{6.21d}$$

Note that

$$\frac{1}{e^2}\,\widetilde{\Sigma}^0_{(\rho)\mathrm{V}}(t,t;\cdot) = 0\,. \tag{6.22}$$

For the spectral components, we obtain:

$$\begin{aligned}
\frac{1}{e^2}\,\Sigma_{(\rho)\mathrm{S}}(t,t';\cdot) = {}& -F^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SS}} \Big[\rho^{(\mathrm{g})}_{\mathrm{S}}(t,t';\cdot) + 2\rho^{(\mathrm{g})}_{\mathrm{T}}(t,t';\cdot) + \rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\Big] \\
& + F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VV}_1} \Big[-\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';\cdot) + \widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';\cdot)\Big] \\
& - \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SS}} \Big[F^{(\mathrm{g})}_{\mathrm{S}}(t,t';\cdot) + 2F^{(\mathrm{g})}_{\mathrm{T}}(t,t';\cdot) + F^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\Big] \\
& + \rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VV}_1} \Big[-\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';\cdot) + \widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';\cdot)\Big]\,,
\end{aligned} \tag{6.23a}$$

$$\begin{aligned}
\frac{1}{e^2}\,\widetilde{\Sigma}^0_{(\rho)\mathrm{V}}(t,t';\cdot) = {}& \widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{SS}} \Big[-\rho^{(\mathrm{g})}_{\mathrm{S}}(t,t';\cdot) + 2\rho^{(\mathrm{g})}_{\mathrm{T}}(t,t';\cdot) + \rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\Big] \\
& + F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VV}_1} \Big[\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';\cdot) + \widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';\cdot)\Big] \\
& + \widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{SS}} \Big[-F^{(\mathrm{g})}_{\mathrm{S}}(t,t';\cdot) + 2F^{(\mathrm{g})}_{\mathrm{T}}(t,t';\cdot) + F^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\Big] \\
& + \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VV}_1} \Big[\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';\cdot) + \widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';\cdot)\Big]\,,
\end{aligned} \tag{6.23b}$$

$$\begin{aligned}
\frac{1}{e^2}\,\Sigma_{(\rho)\mathrm{V}}(t,t';\cdot) = {}& F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VS}} \Big[\rho^{(\mathrm{g})}_{\mathrm{S}}(t,t';\cdot) + 2\rho^{(\mathrm{g})}_{\mathrm{T}}(t,t';\cdot) + \rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\Big] \\
& - 2F^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VT}} \rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot) \\
& + \widetilde{F}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{SV}} \Big[\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';\cdot) + \widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';\cdot)\Big] \\
& + \rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VS}} \Big[F^{(\mathrm{g})}_{\mathrm{S}}(t,t';\cdot) + 2F^{(\mathrm{g})}_{\mathrm{T}}(t,t';\cdot) + F^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\Big] \\
& - 2\rho^{(\mathrm{f})}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{VT}} F^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot) \\
& + \widetilde{\rho}^{(\mathrm{f})0}_{\mathrm{V}}(t,t';\cdot) *_{\mathrm{SV}} \Big[\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';\cdot) + \widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';\cdot)\Big]\,,
\end{aligned} \tag{6.23c}$$

$$
\begin{aligned}
\frac{1}{e^2}\,\Sigma_{(\rho)\mathrm{T}}(t,t';\cdot) = \quad & F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VS}} \left[\rho^{(\mathrm{g})}_{\mathrm{S}}(t,t';\cdot) - 2\rho^{(\mathrm{g})}_{\mathrm{T}}(t,t';\cdot) - \rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\right] \\
& + 2F^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VT}} \rho^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot) \\
& + F^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SV}} \left[-\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';\cdot) + \widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';\cdot)\right] \\
& + \rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VS}} \left[F^{(\mathrm{g})}_{\mathrm{S}}(t,t';\cdot) - 2F^{(\mathrm{g})}_{\mathrm{T}}(t,t';\cdot) - F^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot)\right] \\
& + 2\rho^{(\mathrm{f})}_{\mathrm{T}}(t,t';\cdot) *_{\mathrm{VT}} F^{(\mathrm{g})}_{\mathrm{L}}(t,t';\cdot) \\
& + \rho^{(\mathrm{f})}_{\mathrm{S}}(t,t';\cdot) *_{\mathrm{SV}} \left[-\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';\cdot) + \widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';\cdot)\right]. \qquad (6.23\mathrm{d})
\end{aligned}
$$

Note that

$$
\frac{1}{e^2}\,\widetilde{\Sigma}^0_{(\rho)\mathrm{V}}(t,t;\cdot) = 2F^{(\mathrm{f})}_{\mathrm{V}}(t,t;\cdot) *_{\mathrm{VV}_1} \widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t;\cdot) - 1 *_{\mathrm{SS}} \left[F^{(\mathrm{g})}_{\mathrm{S}}(t,t;\cdot) + 2F^{(\mathrm{g})}_{\mathrm{T}}(t,t;\cdot) + F^{(\mathrm{g})}_{\mathrm{L}}(t,t;\cdot)\right], \qquad (6.24)
$$

while all other equal-time spectral components vanish.

## 6.2 Renormalization

In order to extract "physical" values of quantities, i. e. values which can be measured in an experiment, one has to properly renormalize a QFT. It is well-known how to do this in vacuum; in thermal equilibrium or even out-of-equilibrium, the renormalization program is much more involved [RS06].

There is another problem inherent to gauge theories: A finite momentum (UV) cutoff breaks gauge invariance and induces a finite photon mass.[4]

In fact, it has turned out that in many applications, a full numerical implementation of the renormalization program is not necessary (see, however, Chap. 7). In particular, many results concerning the thermalization of quantum fields have been obtained by studying unrenormalized quantities or quantities where the dominant divergent contributions have been subtracted. These of course depend on the momentum UV cutoff which has to be introduced in order to render quantities finite and to be able to implement the theory numerically in the first place.

The method we employ in numerical calculations in this work is to simply add a mass counterterm in order to account for the most severe divergences (quadratic in the case of the photons and logarithmic in the case of the fermions[5]) and absorb the photon mass induced by the finite momentum cutoff. The mass counterterm has to be independent of time; we therefore calculate it at initial time.

---

[4] Regularization methods exist which respect gauge invariance and therefore preserve the Ward identity, like dimensional regularization. This, however, is not practicable in numerical implementations, where a finite momentum cutoff is the simplest possible regularization method.

[5] This is true in full momentum space. Since the Fourier transformation changes the dimension of quantities, the power of the divergence depends on their representation. For instance, as functions of time and spatial momentum, the dimension is increased by one compared to a pure momentum representation.

### 6.2.1 Photon Mass Counterterm

There is exactly one Lorentz scalar of mass dimension two which can be formed out of the photon propagator, namely its contraction. In the action, this leads to the dimension two mass counterterm[6] [RS06]

$$\int_x \frac{1}{2}\,\delta m^{(\mathrm{g})2} D^{\mu}{}_{\mu}(x,x) = \frac{1}{2}\,\delta m^{(\mathrm{g})2}\,\mathrm{Tr}(D)\,, \tag{6.25}$$

so that the 2PI effective action gets modified according to:

$$\frac{\mathrm{i}}{2}\,\mathrm{Tr}\Big(D_0^{-1}D\Big) \to \frac{\mathrm{i}}{2}\,\mathrm{Tr}\Big(D_0^{-1}D\Big) + \frac{1}{2}\,\delta m^{(\mathrm{g})2}\,\mathrm{Tr}(D) = \frac{1}{2}\,\mathrm{Tr}\Big(\Big(\mathrm{i}\,D_0^{-1} + \delta m^{(\mathrm{g})2}\Big)D\Big)\,.$$

We then define the renormalized photon self-energy $\Pi_{\mathrm{R}}^{\mu\nu}$ as:[7]

$$\begin{aligned}\Pi_{\mathrm{R}}^{\mu\nu}(x,y) &= 2\,\mathrm{i}\,\frac{\delta}{\delta D_{\mu\nu}(x,y)}\Big\{\Gamma_2[D,G] - \frac{\mathrm{i}}{4}\,\delta m^{(\mathrm{g})2}\,\mathrm{Tr}(D)\Big\}\\ &= \Pi^{\mu\nu}(x,y) + \mathrm{i}\,g^{\mu\nu}\delta m^{(\mathrm{g})2}\delta^4(x-y)\,.\end{aligned} \tag{6.26}$$

Since at initial time the theory is free, we can Fourier transform to momentum space, where the renormalized self-energy is given by:

$$\Pi_{\mathrm{R}}^{\mu\nu}(p) = \Pi^{\mu\nu}(p) + \mathrm{i}\,g^{\mu\nu}\delta m^{(\mathrm{g})2}\,. \tag{6.27}$$

Imposing the renormalization condition $\Pi_{\mathrm{R}}^{\mu\nu}(0) = 0$ then fixes the mass counterterm in terms of the unrenormalized photon self-energy to

$$\delta m^{(\mathrm{g})2} = \frac{\mathrm{i}}{4}\,\Pi^{\mu}{}_{\mu}(0)\,. \tag{6.28}$$

The cutoff-regularized perturbative one-loop vacuum photon self-energy reads:

$$\begin{aligned}\Pi^{\mu\nu}(p) &= e^2\int^{\Lambda}\frac{\mathrm{d}^4q}{(2\pi)^4}\,\mathrm{tr}(\gamma^{\mu}S_0(p+q)\gamma^{\nu}S_0(q))\\ &= -e^2\int^{\Lambda}\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\mathrm{tr}\Big(\gamma^{\mu}[\gamma^{\rho}(p_{\rho}+q_{\rho})+m^{(\mathrm{f})}]\gamma^{\nu}(\gamma^{\sigma}q_{\sigma}+m^{(\mathrm{f})})\Big)}{\Big[(p+q)^2 - m^{(\mathrm{f})2} + \mathrm{i}\,\varepsilon\Big](q^2 - m^{(\mathrm{f})2} + \mathrm{i}\,\varepsilon)}\end{aligned} \tag{6.29}$$

[6]This is the only counterterm only in vacuum. Renormalizing at any time later than initial time, when interactions are “turned on”, there are several counterterms. In fact, there are six counterterms with mass dimension $\geq 1$ (two with dimension 2, four with dimension 1), namely (up to a factor 1/2):

$$\delta m_1^{(\mathrm{g})2} D_{00}(x,y)|_{y=x}\,,\qquad \delta m_2^{(\mathrm{g})2} D^i{}_i(x,y)|_{y=x}\,,$$
$$\delta m_3^{(\mathrm{g})}\partial_x^0 D_{00}(x,y)|_{y=x}\,,\quad \delta m_4^{(\mathrm{g})}\partial_x^0 D^i{}_i(x,y)|_{y=x}\,,\quad \delta m_5^{(\mathrm{g})}\partial_x^i D_{i0}(x,y)|_{y=x}\,,\quad \delta m_6^{(\mathrm{g})}\partial_x^i D_{0i}(x,y)|_{y=x}\,.$$

This would then lead to different mass counterterms for the different isotropic components of the photon self-energy.

[7]Note that, since it is local, the counterterm is *not* contained in the 2PI part of the 2PI effective action, but in the “free gas” part.

with the free fermion propagator

$$S_0(p) = \frac{\mathrm{i}(\gamma^\mu p_\mu + m^{(\mathrm{f})})}{p^2 - m^{(\mathrm{f})2} + \mathrm{i}\,\varepsilon} \tag{6.30}$$

and

$$\int^\Lambda \frac{\mathrm{d}^4 q}{(2\pi)^4} = \int_{-\infty}^{\infty} \frac{\mathrm{d}q^0}{2\pi} \int^\Lambda \frac{\mathrm{d}^3 q}{(2\pi)^3}$$

with the spatial momentum cutoff $\Lambda$, i. e. $|\boldsymbol{q}| \leq \Lambda$ for the spatial loop momentum $\boldsymbol{q}$.

Using

$$\begin{aligned}
&\frac{1}{4}\,\mathrm{tr}\Big(\gamma^\mu\Big[\gamma^\rho(p_\rho + q_\rho) + m^{(\mathrm{f})}\Big]\gamma^\nu(\gamma^\sigma q_\sigma + m^{(\mathrm{f})})\Big) \\
&\quad = \frac{1}{4}\Big[\mathrm{tr}(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)(p_\rho + q_\rho)q_\sigma + \mathrm{tr}(\gamma^\mu\gamma^\nu)m^{(\mathrm{f})2}\Big] \\
&\quad = (g^{\mu\rho}g^{\nu\sigma} - g^{\mu\nu}g^{\rho\sigma} + g^{\mu\sigma}g^{\nu\rho})(p_\rho + q_\rho)q_\sigma + g^{\mu\nu}m^{(\mathrm{f})2} \\
&\quad = (p^\mu + q^\mu)q^\nu + (p^\nu + q^\nu)q^\mu + g^{\mu\nu}\Big[m^{(\mathrm{f})2} - (p+q)\cdot q\Big]\,,
\end{aligned}$$

we have:

$$\Pi^{\mu\nu}(p) = -4e^2 \int^\Lambda \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{(p^\mu + q^\mu)q^\nu + (p^\nu + q^\nu)q^\mu + g^{\mu\nu}\Big[m^{(\mathrm{f})2} - (p+q)\cdot q\Big]}{\Big[(p+q)^2 - m^{(\mathrm{f})2} + \mathrm{i}\,\varepsilon\Big](q^2 - m^{(\mathrm{f})2} + \mathrm{i}\,\varepsilon)}\,. \tag{6.31}$$

It follows that

$$\begin{aligned}
\delta m^{(\mathrm{g})2} &= 2\,\mathrm{i}\,e^2 \int^\Lambda \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{q^2 - 2m^{(\mathrm{f})2}}{(q^2 - m^{(\mathrm{f})2} + \mathrm{i}\,\varepsilon)^2} \\
&= 2e^2 \int^\Lambda \frac{\mathrm{d}^3 q_{\mathrm{E}}}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{\mathrm{d}q_{\mathrm{E}}^0}{2\pi} \frac{(q_{\mathrm{E}}^0)^2 + \boldsymbol{q}_{\mathrm{E}}^2 + 2m^{(\mathrm{f})2}}{\Big[(q_{\mathrm{E}}^0)^2 + \boldsymbol{q}_{\mathrm{E}}^2 + m^{(\mathrm{f})2}\Big]^2} \\
&= \frac{e^2}{2} \int^\Lambda \frac{\mathrm{d}^3 q_{\mathrm{E}}}{(2\pi)^3} \frac{2\boldsymbol{q}_{\mathrm{E}}^2 + 3m^{(\mathrm{f})2}}{(\boldsymbol{q}_{\mathrm{E}}^2 + m^{(\mathrm{f})2})^{3/2}} \\
&= \frac{e^2}{8\pi^2} \int_0^\Lambda \mathrm{d}q_{\mathrm{E}}\, q_{\mathrm{E}}^2 \frac{2q_{\mathrm{E}}^2 + 3m^{(\mathrm{f})2}}{(q_{\mathrm{E}}^2 + m^{(\mathrm{f})2})^{3/2}} \\
&= \frac{e^2}{8\pi^2} \frac{\Lambda^3}{\sqrt{\Lambda^2 + m^{(\mathrm{f})2}}} \\
&= \frac{e^2}{8\pi^2} \Lambda^2 \left[1 + \mathcal{O}\left(\frac{m^{(\mathrm{f})2}}{\Lambda^2}\right)\right]\,.
\end{aligned} \tag{6.32}$$

In the photon EoMs one therefore has to replace $p^2 \to p^2 + \delta m^{(\mathrm{g})2}$.

### 6.2.2 Fermion Mass Counterterm

The fermion mass counterterm appearing in the action reads

$$\delta m^{(\mathrm{f})} S(x,x)\,. \tag{6.33}$$

The renormalized fermion self-energy is then given by:

$$\Sigma_{\mathrm{R}}(x,y) = \Sigma(x,y) - \mathrm{i}\,\delta m^{(\mathrm{f})}\delta^4(x-y)\,. \tag{6.34}$$

In Fourier space, the renormalization condition then reads:[8]

$$\Sigma_{\mathrm{R}}(p_*) = 0\,, \tag{6.35}$$

from which it follows that the mass counterterm is given by

$$\delta m^{(\mathrm{f})} = -\frac{\mathrm{i}}{4}\,\Sigma_{\mathrm{S}}(p_*)\,. \tag{6.36}$$

The cutoff-regularized perturbative one-loop vacuum fermion self-energy is given by:

$$\begin{aligned}\Sigma(p) &= -e^2\int^{\Lambda}\frac{\mathrm{d}^4q}{(2\pi)^4}\gamma^\mu S_0(q)\gamma^\nu D_{0\mu\nu}(p-q)\\ &= -e^2\int^{\Lambda}\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{\gamma^\mu(\gamma^\rho q_\rho + m^{(\mathrm{f})})\gamma^\nu}{(q^2-m^{(\mathrm{f})2}+\mathrm{i}\,\varepsilon)\big[(p-q)^2+\mathrm{i}\,\varepsilon\big]}\left[g_{\mu\nu}-(1-\xi)\frac{(p_\mu-q_\mu)(p_\nu-q_\nu)}{(p-q)^2}\right]\end{aligned} \tag{6.37}$$

Its scalar component then reads:

$$\begin{aligned}\Sigma_{\mathrm{S}}(p) &= \frac{1}{4}\,\mathrm{tr}(\Sigma_{\mathrm{S}}(p))\\ &= -e^2m^{(\mathrm{f})}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{g^{\mu\nu}}{(q^2-m^{(\mathrm{f})2}+\mathrm{i}\,\varepsilon)\big[(p-q)^2+\mathrm{i}\,\varepsilon\big]}\left[g_{\mu\nu}-(1-\xi)\frac{(p_\mu-q_\mu)(q_\nu-q_\nu)}{(p-q)^2}\right]\\ &= -(3+\xi)\,e^2m^{(\mathrm{f})}\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(q^2-m^{(\mathrm{f})2}+\mathrm{i}\,\varepsilon)\big[(p-q)^2+\mathrm{i}\,\varepsilon\big]}\,.\end{aligned} \tag{6.38}$$

We will do the calculation for two renormalization points, $p_*^2 = 0$ and $p_*^2 = m^{(\mathrm{f})2}$, and show that the resulting mass counterterm is independent of the renormalization point used to calculate it.

[8]Note that, due to Lorentz invariance, $\Sigma_{\mathrm{R}}(p_*) = \Sigma_{\mathrm{R}}(p)|_{p^2=p_*^2}$.

$\underline{p_*^2 = 0}$ It follows that

$$
\begin{aligned}
\delta m^{(\mathrm{f})}|_{p^2=0} &= -\frac{\mathrm{i}}{4}\,\Sigma_{\mathrm{S}}(0) \\
&= \frac{\mathrm{i}(3+\xi)}{4}\,e^2 m^{(\mathrm{f})} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2 - m^{(\mathrm{f})2} + \mathrm{i}\,\varepsilon)(q^2 + \mathrm{i}\,\varepsilon)} \\
&= -\frac{3+\xi}{4}\,e^2 m^{(\mathrm{f})} \int \frac{\mathrm{d}^4 q_{\mathrm{E}}}{(2\pi)^4} \frac{1}{(q_{\mathrm{E}}^2 + m^{(\mathrm{f})2})\,q_{\mathrm{E}}^2} \\
&= -\frac{3+\xi}{4}\,e^2 m^{(\mathrm{f})} \int \frac{\mathrm{d}^3 q_{\mathrm{E}}}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{\mathrm{d}q_{\mathrm{E}}^0}{2\pi} \frac{1}{[(q_{\mathrm{E}}^0)^2 + \boldsymbol{q}_{\mathrm{E}}^2 + m^{(\mathrm{f})2}][(q_{\mathrm{E}}^0)^2 + \boldsymbol{q}_{\mathrm{E}}^2]} \\
&= -\frac{(3+\xi)\,e^2}{8}\,m^{(\mathrm{f})} \int \frac{\mathrm{d}^3 q_{\mathrm{E}}}{(2\pi)^3} \frac{1}{|\boldsymbol{q}_{\mathrm{E}}|\Big[|\boldsymbol{q}_{\mathrm{E}}|\Big(\sqrt{\boldsymbol{q}_{\mathrm{E}}^2 + m^{(\mathrm{f})2}} + |\boldsymbol{q}_{\mathrm{E}}|\Big) + m^{(\mathrm{f})2}\Big]} \\
&= -\frac{(3+\xi)\,e^2}{32\pi^2}\,m^{(\mathrm{f})} \int_0^{\Lambda} \mathrm{d}q_{\mathrm{E}} \frac{q_{\mathrm{E}}}{q_{\mathrm{E}}\Big(\sqrt{q_{\mathrm{E}}^2 + m^{(\mathrm{f})2}} + q_{\mathrm{E}}\Big) + m^{(\mathrm{f})2}} \\
&= -\frac{(3+\xi)\,e^2}{32\pi^2}\,m^{(\mathrm{f})} \frac{\Lambda\Big(\Lambda - \sqrt{\Lambda^2 + m^{(\mathrm{f})2}}\Big) + m^{(\mathrm{f})2}\ln\Big(\frac{\Lambda + \sqrt{\Lambda^2 + m^{(\mathrm{f})2}}}{m^{(\mathrm{f})}}\Big)}{2m^{(\mathrm{f})2}} \\
&= -\frac{(3+\xi)\,e^2}{64\pi^2}\,m^{(\mathrm{f})} \left[\ln\left(\frac{\Lambda + \sqrt{\Lambda^2 + m^{(\mathrm{f})2}}}{m^{(\mathrm{f})}}\right) + \frac{\Lambda\Big(\Lambda - \sqrt{\Lambda^2 + m^{(\mathrm{f})2}}\Big)}{m^{(\mathrm{f})2}}\right] \\
&\xrightarrow{\Lambda\to\infty} -\frac{(3+\xi)\,e^2}{64\pi^2}\,m^{(\mathrm{f})} \left[\ln\left(\frac{\Lambda}{m^{(\mathrm{f})}}\right) + \mathcal{O}\left(\frac{m^{(\mathrm{f})2}}{\Lambda^2}\right)\right]\,.
\end{aligned}
\tag{6.39}
$$

$\underline{p_*^2 = m^{(\mathrm{f})2}}$ Using

$$
(p-q)^2|_{p^2=m^{(\mathrm{f})2}} = (p^2 - 2p\cdot q + q^2)|_{p^2=m^{(\mathrm{f})2}} = m^{(\mathrm{f})2} - 2m^{(\mathrm{f})}\,q^0 + q^2\,,
$$

we have:

$$
\begin{aligned}
\delta m^{(\mathrm{f})} &= -\frac{\mathrm{i}}{4}\,\Sigma_{\mathrm{S}}(p)|_{p^2=m^{(\mathrm{f})2}} \\
&= \frac{\mathrm{i}(3+\xi)}{4}\,e^2 m^{(\mathrm{f})} \int \frac{\mathrm{d}^4 q}{(2\pi)^4} \frac{1}{(q^2 - m^{(\mathrm{f})2} + \mathrm{i}\,\varepsilon)(m^{(\mathrm{f})2} - 2m^{(\mathrm{f})}q_0 + q^2 + \mathrm{i}\,\varepsilon)} \\
&= -\frac{3+\xi}{4}\,e^2 m^{(\mathrm{f})} \int \frac{\mathrm{d}^4 q_{\mathrm{E}}}{(2\pi)^4} \frac{1}{(q_{\mathrm{E}}^2 + m^{(\mathrm{f})2})(q_{\mathrm{E}}^2 + 2\,\mathrm{i}\,m^{(\mathrm{f})}q_{\mathrm{E}}^0 - m^{(\mathrm{f})2})} \\
&= -\frac{3+\xi}{4}\,e^2 m^{(\mathrm{f})} \int \frac{\mathrm{d}^3 q_{\mathrm{E}}}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{\mathrm{d}q_{\mathrm{E}}^0}{2\pi} \frac{1}{\Big[(q_{\mathrm{E}}^0)^2 + \boldsymbol{q}_{\mathrm{E}}^2 + m^{(\mathrm{f})2}\Big]\Big[(q_{\mathrm{E}}^0)^2 + 2\,\mathrm{i}\,m^{(\mathrm{f})}q_{\mathrm{E}}^0 + \boldsymbol{q}_{\mathrm{E}}^2 - m^{(\mathrm{f})2}\Big]} \\
&= -\frac{(3+\xi)\,e^2}{16}\,m^{(\mathrm{f})} \int \frac{\mathrm{d}^3 q_{\mathrm{E}}}{(2\pi)^3} \frac{1}{\boldsymbol{q}_{\mathrm{E}}^2\sqrt{\boldsymbol{q}_{\mathrm{E}}^2 + m^{(\mathrm{f})2}}}\,\Theta(|\boldsymbol{q}_{\mathrm{E}}| - m^{(\mathrm{f})})
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{(3+\xi)\,e^2}{64\pi^2}\,m^{(f)} \int_0^{\Lambda} \mathrm{d}q_{\mathrm{E}}^0 \,\frac{1}{\sqrt{q_{\mathrm{E}}^2 + m^{(f)2}}}\,\Theta(q_{\mathrm{E}} - m^{(f)}) \\
&= -\frac{(3+\xi)\,e^2}{64\pi^2}\,m^{(f)} \int_{m^{(f)}}^{\Lambda} \mathrm{d}q_{\mathrm{E}}^0 \,\frac{1}{\sqrt{q_{\mathrm{E}}^2 + m^{(f)2}}} \\
&= -\frac{(3+\xi)\,e^2}{64\pi^2}\,m^{(f)} \ln\left(\frac{(\sqrt{2}-1)\left(\Lambda + \sqrt{\Lambda^2 + m^{(f)2}}\right)}{m^{(f)}}\right) \\
&= -\frac{(3+\xi)\,e^2}{64\pi^2}\,m^{(f)} \left[\ln\left(\frac{\Lambda + \sqrt{\Lambda^2 + m^{(f)2}}}{m^{(f)}}\right) + \ln\left(\sqrt{2}-1\right)\right] \\
&\xrightarrow{\Lambda\to\infty} -\frac{(3+\xi)\,e^2}{64\pi^2}\,m^{(f)} \left[\ln\left(\frac{\Lambda}{m^{(f)}}\right) + \mathcal{O}\left(\frac{m^{(f)2}}{\Lambda^2}\right)\right] .
\end{aligned}
\tag{6.40}
$$

Asymptotically, i. e. for $m^{(f)}/\Lambda \to 0$, the value of the mass counterterm is therefore independent of the renormalization point, as expected.

## 6.3 Results

The numerical solution of the 2PI EOMs is very challenging. In fact, a fourth-order Runge–Kutta (RK4) method has to be employed only to solve the *free* photon EOMs in order to get a stable evolution. Further, in general very small time steps have to be used to evolve the equations.

All the results presented in the following have been generated with a thermal initial distribution function for the photons and a Gaussian initial distribution function for the fermions:

$$
n_0^{(g)}(p) = \frac{1}{\mathrm{e}^{p/T} - 1}\,, \qquad n_0^{(f)}(p) = A \exp\left(-\frac{(p-p_0)^2}{2\sigma^2}\right)
$$

with $T = m^{(f)}$ and $(A, p_0, \sigma) = (0.1, 0.3\,m^{(f)}, 2\,m^{(f)})$. Further, the gauge fixing parameter is $\xi = 0.5$.

We start by presenting the numerical solution for the photon statistical function, for both the free and the interacting case, in Fig. 6.4. A plot of the analytical solution to the free photon EOMs would lie almost exactly on top of the numerical one, i. e. the free photon EOMs are solved to a very good accuracy.

In the interacting case, there are two interesting things to notice: First, the interacting solutions grow even stronger than the analytical ones. Although, as we have argued in Sec. 4.2, it is to be expected that the interacting solutions are not secular, so that at some point the growth would have to stop and a damping should set in, for those times which are accessible for us this is not visible, and there are no indications that the growth will stop at some point.

And second, there are rather large deviations of the interacting solutions from the free ones. For such a weakly coupled theory, this is not what one would expect. The large deviations seem to imply that this might be a numerical problem. After all, as mentioned

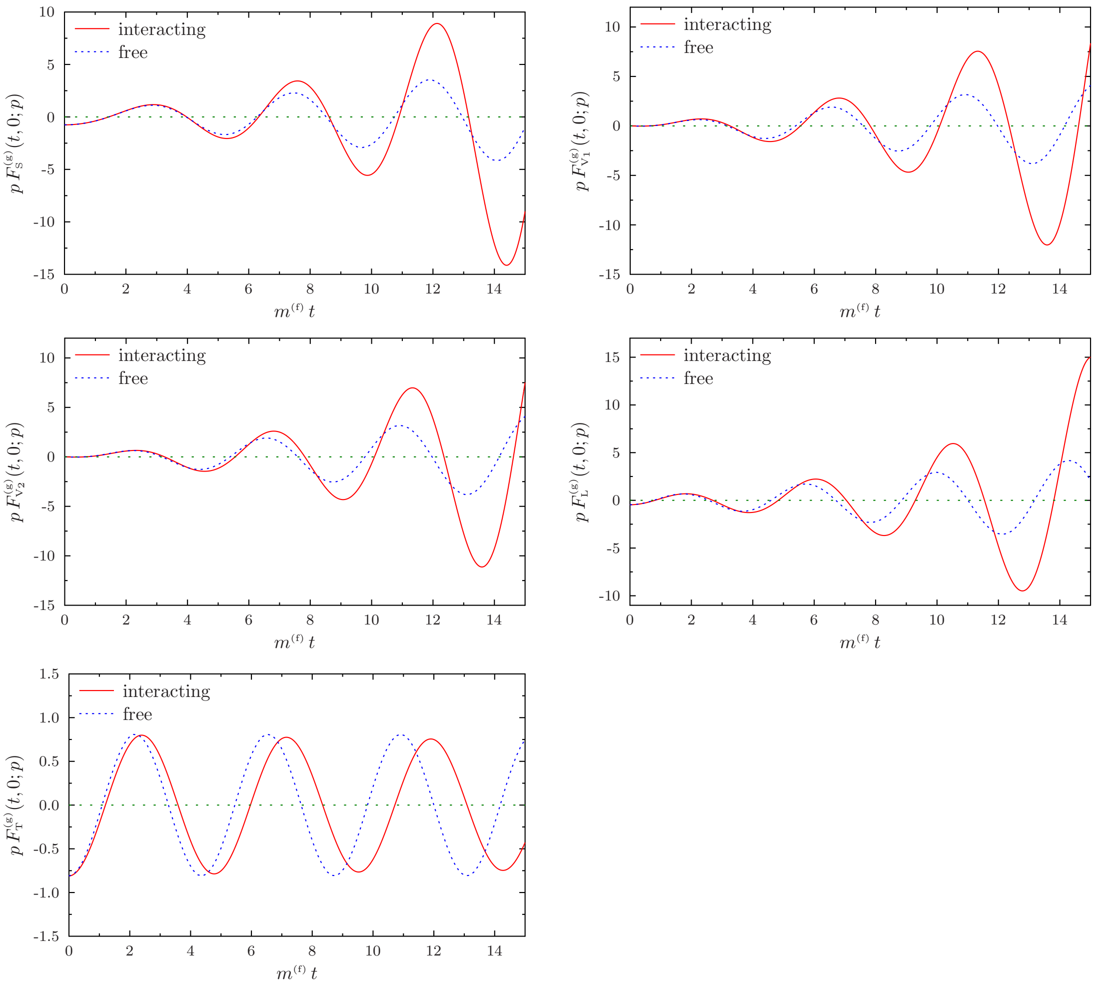



Figure 6.4: The isotropic components of the photon statistical function for $p = \frac{11}{24}\,\pi\,m^{(\mathrm{f})}$.

before, already for the solution of the free photon EoMs, a fourth-order Runge–Kutta method is needed, and even that might fail in the interacting case so that even more sophisticated methods might have to be employed. This is a clear sign that although the question of the secularity of the photon correlation functions might not be of principle relevance, it most certainly is of practical relevance.

In contrast to the solutions for the photon correlation functions, the solutions for the fermion correlation functions shown in Fig. 6.5 look at least qualitatively correct. There is a light damping, as expected, and the frequency in the interacting case is increased compared to the free case, which hints at the generation of a "thermal" mass[9]. The tensor component, which in thermal equilibrium would be exactly zero, is not exactly zero, but

[9]Since we start with a nonequilibrium initial state, it would be more correct to say "a mass generated dynamically from the interactions with the system".

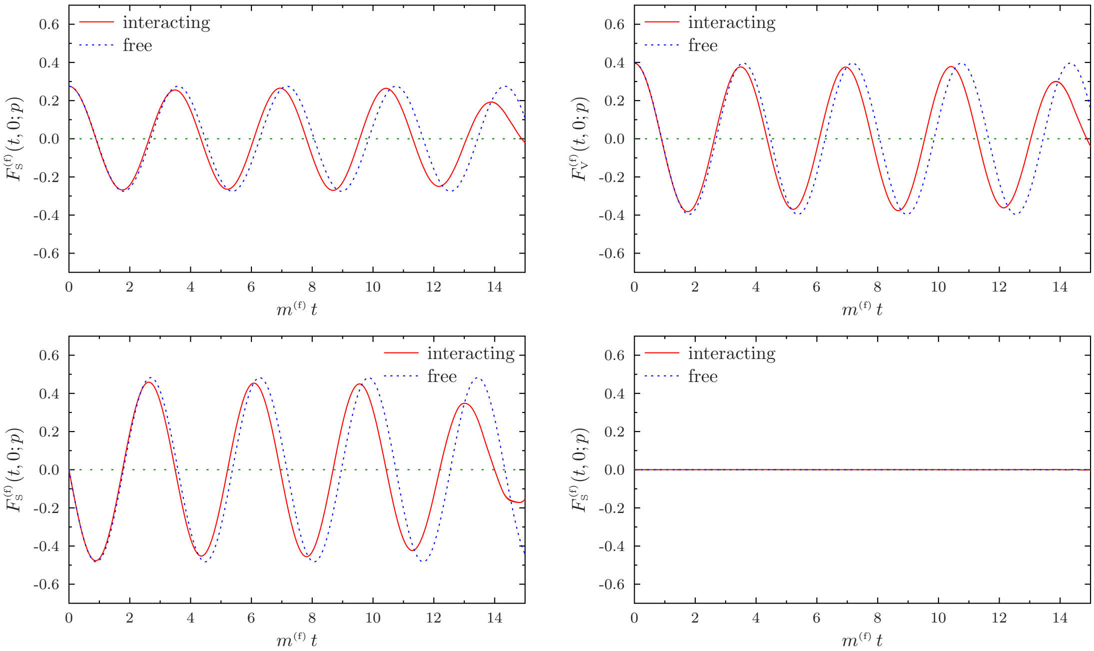



Figure 6.5: The Lorentz components of the fermion statistical function for $p = \frac{11}{24}\,\pi\, m^{(f)}$.

so close to it that it cannot be distinguished from the free solution.

Finally, in Fig. 6.6 we show the isotropic components of the resummed and perturbative statistical photon self-energies.[10] It is interesting to note that they are damped away almost completely even for the small times displayed. That the resummed and perturbative self-energies are hardly distinguishable is a further sign for the very weak coupling of QED.

[10]The resummed self-energies are those containing the full propagators, while the perturbative self-energies are calculated with the free propagators.

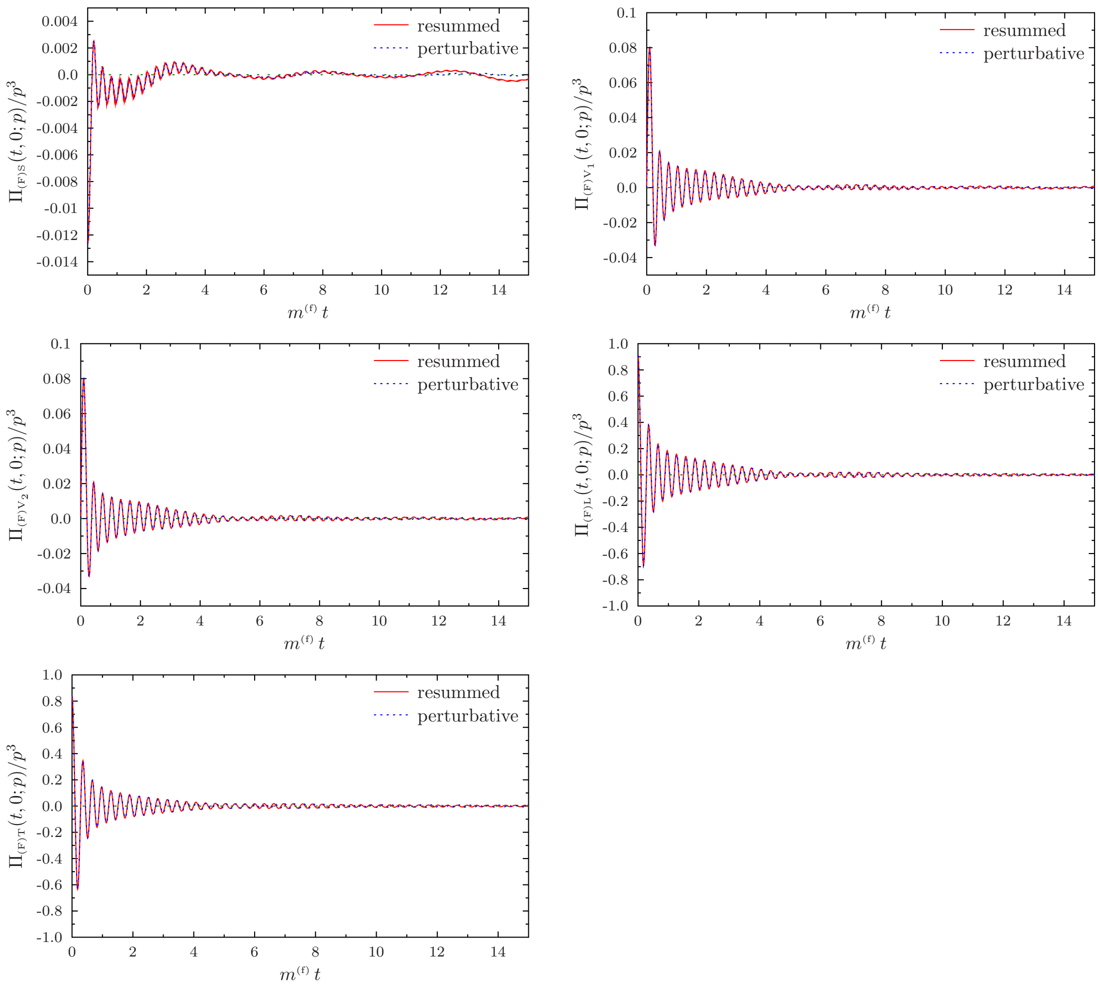



Figure 6.6: The isotropic components of the resummed and perturbative photon statistical self-energy for $p = \frac{11}{24}\,\pi\, m^{(f)}$.

# Chapter 7

# Conclusions and Outlook

In this work, we have discussed the real-time formulation of abelian gauge theories, with the physical example of QED, within the 2PI framework. Although the derivation of the EOMs for the two-point correlation functions, which contain important information if one is interested e. g. in questions regarding the thermalization of a theory, from the 2PI effective action is straightforward, a closer examination of their structure reveals subtleties characteristic for gauge theories. These subtleties are connected to the fact that a Lorentz covariant formulation of gauge theories necessitates the introduction of unphysical, redundant DOFs. While in vacuum or in thermal equilibrium it is usually possible to project onto the physical DOFs in the first place, this is not the case out-of-equilibrium, where time-translation invariance is lost. This prohibits the formulation of (local) projection operators in practice. It is therefore practically unavoidable to evolve unphysical DOFs as well. A remarkable property of the solution to the free EOMs of those unphysical DOFs is that they diverge in time, i. e. are secular for most gauges. This behavior does not indicate a failure of the formulation of the theory, though: Since these DOFs are not physical in the first place, they need not be bounded. These secularities are of practical relevance since handling diverging quantities or quantities which at least acquire large values is problematic from a numerical point of view.

The EOMs for the photon two-point functions in linear covariant gauges (except for Feynman gauge, which constitutes a special case) are structurally rather complicated. This is due to the gauge fixing parameter dependent part of the kinetic term which is "non-diagonal" and therefore mixes different components of the gradient. This in turn leads to EOMs which are not purely second-order in time, but contain first derivatives as well. These features make solving the EOMs numerically challenging. We have therefore presented a reformulation of the photon EOMs which employs an auxiliary field, the so-called Nakanishi–Lautrup field. Here another interesting feature presented itself: On first sight, the auxiliary field seems to be noninteracting since in the EOMs for the auxiliary field correlation functions, the respective memory integrals exclusively depend on the longitudinal part of the photon self-energy. But the longitudinal part of the photon self-energy is known from the Ward identities to vanish identically. If this were indeed the case, the EOMs for the auxiliary correlators could be solved analytically (after all, they correspond to mere harmonic oscillators), and their solution could be plugged back into the EOM for the correlation function involving only photon fields (the original photon EOM), thereby getting rid of the auxiliary field correlators altogether (effectively "integrating them out"). It turns out that the resulting EOMs for the photon correlators are structurally much simpler than the original ones. In particular, the resulting equations are purely second-order in time, and the explicit dependence on the gauge fixing parameter is linear (in contrast to the original EOMs which contain the *inverse* gauge fixing parameter)[1], so that it is obvious that the limit of Landau gauge is in fact well-defined.

However, the photon self-energy involved in the EOMs is in fact *not* transverse if derived from a finitely truncated 2PI effective action. This is related to the complex resummation scheme implemented in the 2PI effective action which mixes different perturbative orders, and the nontransversality was shown explicitly in an analytic way for the case of the two-loop truncation employed in the numerics.[2] Therefore, there is the remarkable situation that for any finite truncation of the 2PI effective action, the EOMs for the auxiliary correlators are *not* free, although they are in the exact theory. From a practical point of view, where one always has to work with finite truncations, this means nothing has been gained from the reformulation of the EOMs. In fact, since we have increased the number of EOMs (since in addition to the EOMs for the purely photonic correlation functions, one also has to solve the EOMs for the auxiliary correlation functions), we have even complicated the situation. That the reformulation of the photon EOMs is useful and instructive nevertheless is due to the fact that their structure reveals the origin of the secularities mentioned above.[3] Besides, it allows for an easy solution of the free EOMs, which is not quite as obvious from the original EOMs.

From the reformulated photon EOMs, it can easily be seen that the secularities stem from a peculiar resonance effect which occurs for all linear covariant gauges except for Feynman gauge (in which case the free EOMs—the original ones as well as the reformulated ones, which are identical in this gauge—just describe simple harmonic oscillators). An interesting question is whether this resonance is artificial in the sense that it only occurs in the free theory, or if it persists even in the full theory. Since it is unlikely that this question can be answered from an analytical consideration (due to the complexity of the EOMs), one has to resort to numerical methods. In a numerical simulation, however, sufficiently late times in order to answer this question are so far out of reach. It is nevertheless improbable that the secularities do persist in the full theory since most likely, the interactions will push the system away from the resonance. Due to the weak coupling of QED, however, it is to be expected that significant deviations from the free theory will occur only at rather late times, which means that the correlation functions will probably grow for a long time until some sort of damping sets in which renders their solutions finite. This is at least a practical

[1] Due the fact that the *full* propagators appear in the EOMs, there are implicit dependencies on the gauge fixing parameter though.

[2] Of course, the photon self-energy *is* transverse in the full theory, where *all* perturbative orders contribute.

[3] And, in a more subtle way, to demonstrate the importance of the Ward identities in gauge theories, or rather the problems which show up if they cannot be applied.

problem for numerical simulations, since handling large quantities is often complicated on a computer.

We then turned to another potential problem of gauge theories within the 2PI framework: That of gauge dependencies. Due to the complex resummation the 2PI effective action implements, questions regarding gauge dependencies (i. e. dependencies on the gauge fixing parameter $\xi$) are usually more involved than in perturbation theory. For instance, it is not true in general that correlation functions derived from the 2PI effective action satisfy the Ward identities. Further, quantities which are gauge invariant in the exact theory or at each perturbative order need not be gauge invariant in a finitely truncated 2PI effective action. Although it can be shown that the gauge dependent terms are always of higher order in the coupling than the truncation of the 2PI effective action (i. e. of $\mathcal{O}(e^6)$ for the two-loop truncation of the 2PI effective action considered in this work), this is nevertheless a potential problem in real-time formulations where the expansion parameter is not the coupling constant itself, but the coupling constant multiplied by time. Obviously, this is a serious issue which one certainly has to face if one is interested in late-time physics like thermalization.

For the sake of definiteness, a concrete approximation of the 2PI effective action for QED was considered, namely a two-loop truncation. Since for a finitely truncated 2PI effective action, the reformulation of the photon EOMs is not practical (since the auxiliary field correlators are not free and would hence have to be solved numerically as well, thereby increasing the number of equations to be solved, as already mentioned above), we solve the original photon EOMs numerically. In particular, the self-energies in terms of the full propagators were presented for the approximation employed. Further, the photon and fermion mass counterterm were computed. Together with the EOMs, one then in principle has all the ingredients necessary for solving the system numerically on a computer.

Solving the 2PI EOMs on a computer is challenging, though. This is because of the structural complexity of the photon EOMs and the large number of components which have to be evolved. It is even difficult only to obtain numerical stability for the accessible times. In general, one has to use very small time steps which makes it difficult to reach late times. For certain questions it might also turn out to be necessary to fully implement the renormalization program laid out in Ref. [RS06].

In conclusion, there remain many problems and open questions regarding the real-time formulation of gauge theories within the 2PI framework, making it an exciting subject to work on. Some of them are:

- **The numerical implementation.** The numerical implementation is very delicate, and it is hard to assess the numerical stability and reliability of the results. This can for instance be seen at the fact that (in contrast to, for instance, scalar theories, where a simple Euler method is sufficient) a fourth-order Runge–Kutta method has to be employed in order to only solve the *free* photon EOMs numerically. It also seems to be a general rule that much smaller time steps have to be used as compared to e. g. scalar or fermionic theories. This is probably due to the complicated structure of the photon EOMs and the many coupled components which have to be evolved at

the same time.
For most interesting questions, the accessible times also pose a strong limitation, and the late-time behavior, which should eventually lead to a thermalized system, is at present practically unaccessible. Since QED is very weakly coupled theory, significant deviations from the free theory set in rather late. Moving away from physical QED and increasing the coupling constant is unfortunately also limited since we employ a loop expansion of the 2PI effective action.
Further, there is the question of renormalization. In our numerics, we either work with unrenormalized quantities or employ photon and fermion mass counterterms at most. Although the full renormalization program for abelian gauge theories has been elaborated in great detail in [RS06], it is challenging to implement. Implementing the full renormalization program numerically, however, would open up the possibility to study questions regarding gauge fixing dependencies numerically.
A further question affects already the regularization of the theory. For practical reasons, we employ a simple cutoff regularization in our numerics. Such a regularization, however, violates gauge invariance and thereby effectively introduces a finite photon mass, which is of course unphysical.

- **The physical DOFs**. In a spatially isotropic, homogeneous system as we consider, there exist two photon DOFs (instead of one in the vacuum)[4]: One corresponds to the usual transverse photon (the "fundamental" photon appearing in the action), while the other one, the so-called *plasmon*, is longitudinal and corresponds to a collective DOF which only exists in a medium and emerges due to its interaction with the medium. While in a medium which is in thermal equilibrium, one can easily project the respective quantities onto those two DOFs, this is not easily possible out-of-equilibrium, for reasons mentioned earlier.
- **Gauge dependencies**. Since the expected gauge dependencies can be inferred from analytical considerations, it would be interesting to see if they are observed numerically as well. This question is related to the above mentioned question of renormalization, since high accuracy and properly renormalized quantities are probably needed in order to answer this question.
  Since in real-time formulations of QFTs, the expansion parameter is the product of coupling and time, one expects potential problems at sufficiently late times since gauge dependent terms may grow to values which cannot be neglected.
- **The choice of gauge**. Is there a "best" gauge to work in in the real-time formulation of gauge theories? While a huge body of work indicates that Landau gauge is very convenient in many questions related to gauge theories in vacuum [vSHA98, vSAH97, AvS01], it seems likely that Feynman gauge is a very convenient choice in the real-time formulation of gauge theories. This is because the photon EOMs become structurally much simpler in this case, and in particular the DOFs one has to evolve are not secular. This avoids many potential problems one might have to face in other gauges.

[4]We do not count the two degenerate transverse DOFs separately.

It may be even better to work in noncovariant gauges. Coulomb gauge is very appealing due to its physical nature, i. e. the fact that it contains only the physical DOFs in the first place. Unfortunately, it is not even clear what the photon EOMs look like in this case.[5] Another candidate for a noncovariant gauge is temporal axial gauge, which has proved useful in classical statistical simulations of gauge theories [BSS08, BSS09, BGSS09].

- **Thermalization.** The question of whether it can be shown numerically that a theory thermalizes is one of the main applications of the 2PI formalism. Thermalization is a late-time phenomenon, however, and therefore challenging to observe in a numerical simulation. This is even more so for QED since it is a very weakly coupled theory which implies large thermalization times.
  It would, however, be exciting to see how QED thermalizes starting from a nonequilibrium initial state since due to the structural difference of gauge theories as compared to non-gauge theories, one can expect a qualitatively different process as for other theories for which thermalization has been demonstrated numerically.
- **Comparison with thermal quantities.** It would be interesting to compare thermal quantities extracted from a time evolution from the 2PI EOMs with known results from calculations carried out for thermal equilibrium. One example for such a quantity is the damping rate of a particle propagating in a medium. This is, however, complicated by the fact that a thermal state is not Gaussian (so that one could at best start from an initial state which is close to thermal equilibrium, and it is usually not easy to tell how close exactly one is to equilibrium). As mentioned in the last point of this list, it is in fact possible to implement non-Gaussian, and in particular thermal, states in the 2PI framework, but this is rather involved.
- **Non-abelian gauge theories.** If the questions stated above are answered and a thorough understanding of the real-time formulation of abelian gauge theories is achieved: Can the study be extended to non-abelian gauge theories without running into new conceptual problems related to the real-time formulation of such theories? An alternative to studying an *abelian* gauge theory *including* fermions (which would be trivial without fermions, of course) would then be to study a *non-abelian* gauge theory *without* fermions. A numerical study of a pure SU(2) gauge theory in $2 + 1$ dimensions in temporal axial gauge (i. e. in a noncovariant gauge) has been carried out in Ref. [NO11]. This work has to be taken with a grain of salt, however, since it relies on several assumptions. For instance, the spatially longitudinal DOF which emerges in a medium, i. e. the plasmon, is discarded altogether. Further, a thermal mass is inserted by hand for practical reasons instead of being generated dynamically in the evolution of the system.
- **Non-Gaussian initial states.** What is the effect of non-Gaussian initial states? Although Gaussian initial states suggest themselves to be used in approaches based

[5] It results from a limiting procedure in a way similar to Landau gauge in the class of covariant gauges; one would therefore probably have to introduce some sort of “generalized Coulomb gauge” parametrized by some parameter like for the covariant gauges.

on the 2PI effective action, it is in fact possible to implement non-Gaussian initial states as well in the 2PI framework. Although rather involved, it has for instance been shown in Ref. [GM09] how to implement a thermal initial state.

In conclusion, there are lots of open questions regarding gauge theories in the 2PI framework, even in the allegedly simple abelian case. This work has laid the foundation for answering some of the above mentioned questions, but much more work has to be done in order to understand gauge theories on a level comparable to scalar or fermionic theories.

# Appendix A

# The Nakanishi–Lautrup Field in the Operator Formalism

It is useful to shortly consider the formulation of QED including the NL field in the operator formalism, since it allows for an easy derivation of certain identities which hold in the exact theory.[1] For an in-depth treatment of this issue, see e. g. the monograph [NO90].

In the operator formalism, the fact that $B$ is an auxiliary field translates to the fact that it can be expressed in terms of the photon field (corresponding to the fact that it can be integrated out in the path integral). The operator EOMs following from the action $S_{\mathrm{NL}}[A,B]+S_{\mathrm{f}}[\psi,\overline{\psi}]+S_{\mathrm{int}}[A,\psi,\overline{\psi}]$ are given by

$$\partial^\mu F_{\mu\nu} = \partial_\nu B + J_\nu \qquad \text{(“quantum Maxwell equation”)}, \tag{A.1a}$$

$$B = -\frac{1}{\xi}\,\partial^\mu A_\mu\,, \tag{A.1b}$$

with the current $J^\mu = e\,\overline{\psi}\,\gamma^\mu\psi$. It follows that the original action (2.2) and the NL action (3.52) are equal on-shell, i. e. if the EOMs are satisfied, since $S_B[A,-\partial^\mu A_\mu/(2\xi)] = S_{\mathrm{gf}}[A]$. From the antisymmetry of $F_{\mu\nu}$ it further follows that

$$0 = \partial^\mu\partial^\nu F_{\mu\nu} = \Box B + \partial_\mu J^\mu\,,$$

i. e., provided that the current is conserved, $\partial_\mu J^\mu = 0$, $B$ is a free massless scalar field.[2]

As was mentioned several times before in this work, it is one of the characteristic features of covariant formulations of gauge theories that they contain unphysical DOFs. In

[1] They will *not* hold, in general, in a finitely truncated theory, and since there is no easy connection to the variational correlation functions derived from the 2PI effective action, these identities cannot easily be carried over to the path integral formulation.

[2] This is *only* true for an abelian gauge theory. For a non-abelian gauge theory with field strength $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^a{}_{bc} A^b_\mu A^c_\nu$ (with gauge coupling $g$ and structure constants $f^a{}_{bc}$ of the corresponding gauge group), we have $\partial^\mu\partial^\nu F^a_{\mu\nu} = -g f^a{}_{bc}\partial^\mu\partial^\nu(A^b_\mu A^c_\nu) = -g f^a{}_{bc}\big[\xi^2 B^b B^c - \xi(A^c_\mu\partial^\mu B^b + A^b_\mu\partial^\mu B^c) + (\partial^\nu A^b_\mu)(\partial^\mu A^c_\nu)\big] = -g f^a{}_{bc}(\partial^\nu A^b_\mu)(\partial^\mu A^c_\nu)$, i. e. the EOM for $B$ would be $\Box B^a + g f^a{}_{bc}(\partial^\nu A^b_\mu)(\partial^\mu A^c_\nu) = 0$: In a non-abelian gauge theory, the NL field couples to the gauge field.

operator language, this translates to the fact that the Hilbert space considered is too large and contains the physical Hilbert space as a subspace. The importance of the NL formalism lies in the fact that it can be used to define the physical subspace $\mathcal{V}_{\text{phys}} \subset \mathcal{V}$ of the Hilbert space $\mathcal{V}$ of the quantized theory: A physical state $|f\rangle \in \mathcal{V}_{\text{phys}}$ is a state which is annihilated by the positive-frequency part $B^{(+)}(x)$ of the NL field (since it is free, as we have just shown, we can always decompose the NL field into a positive and a negative frequency part), i. e.

$$B^{(+)}(x)|f\rangle = 0\,. \tag{A.2}$$

This is the *Gupta subsidiary condition.*[3] One direct consequence is that the expectation value of the NL field with respect to physical states vanishes, $\langle B(x)\rangle = 0$. The expectation values of the EoMs (A.1) hence read:

$$\langle \partial^\mu F_{\mu\nu}\rangle = \langle J_\nu\rangle\,, \tag{A.3a}$$

$$\langle \partial^\mu A_\mu\rangle = 0\,, \tag{A.3b}$$

i. e. the Maxwell equation holds in the physical subspace, and the Lorentz gauge condition is valid for the expectation value of gauge field operators in *any* linear covariant gauge, but only in Landau gauge it is valid for the gauge field operators themselves, i. e. as an operator equation.[4]

What we would like to have is a similar condition for correlation functions. But $\langle B(x)\rangle = 0$ does of course *not* imply that $\langle B(x)\, A_{\mu_1}(y_1)\dots A_{\mu_n}(y_n)\rangle = 0$. It does, however, follow from the EoM of $B$ that

$$\Box_x \langle B(x)\, A_{\mu_1}(y_1)\dots A_{\mu_n}(y_n)\rangle = 0\,.$$

Similarly,

$$\partial_x^\mu \langle A_\mu(x) A_{\nu_1}(y_1)\dots A_{\nu_n}(y_n)\rangle = -\xi\,\langle B(x) A_{\nu_1}(y_1)\dots A_{\nu_n}(y_n)\rangle\,.$$

The important point now is to note that

$$\partial_x^\mu \rho^{(\mathrm{g})}_{\mu\nu}(x,y) = \mathrm{i}\langle[\partial_x^\mu A_\mu(x), A_\nu(y)]\rangle = -\,\mathrm{i}\,\xi\,\langle[B(x), A_\nu(y)]\rangle = -\xi\,\rho^{(BA)}_\nu(x,y)\,, \tag{A.4}$$

which follows from the operator EoM (A.1b). Employing this identity, we can rewrite the kinetic term, i. e. the left-hand side, of the EoM (3.21e) as

$$\left[g_\mu{}^\lambda \Box_x - \left(1-\frac{1}{\xi}\right)\partial_{x\mu}\partial_x^\lambda\right]\rho^{(\mathrm{g})}_{\lambda\nu}(x,y) = \Box_x \rho^{(\mathrm{g})}_{\mu\nu}(x,y) + (\xi-1)\,\partial_{x\mu}\rho^{(BA)}_\nu(x,y)\,. \tag{A.5}$$

[3]It extends the Gupta-Bleuler condition [Gup50, Ble50] which, before the NL formalism was developed, was used to quantize QED in the Feynman gauge, and is essentially a statement about the gauge invariance of the physical subspace.

[4]For $\xi = 0$ (i. e. Landau gauge), the NL field is hence nothing but a Lagrange multiplier enforcing $\partial^\mu A_\mu = 0$.

The EoMs for the spectral functions are then given by:

$$\Box_x \rho^{(\mathrm{g})}_{\mu\nu}(x,y) = (1-\xi)\,\partial_{x\mu}\rho^{(BA)}_{\nu}(x,y) + I^{(\mathrm{g})}_{(\rho)\mu\nu}(x,y)\,, \tag{A.6a}$$
$$\Box_x \rho^{(AB)}_{\mu}(x,y) = 0\,, \tag{A.6b}$$
$$\Box_x \rho^{(BA)}_{\mu}(x,y) = 0\,, \tag{A.6c}$$
$$\Box_x \rho^{(BB)}(x,y) = 0\,, \tag{A.6d}$$

where $I^{(\mathrm{g})}_{(\rho)\mu\nu}(x,y)$ is the (spectral) memory integral.[5] The EoMs for correlators involving an auxiliary field are therefore always free.

Fourier transforming with respect to space, the equations read:

$$\left(\frac{\partial^2}{\partial t^2} + \boldsymbol{p}^2\right)\rho^{(\mathrm{g})}_{\mu\nu}(t,t';\boldsymbol{p}) = (1-\xi)\left(\delta^0_\mu\frac{\partial}{\partial t} - \mathrm{i}\,\delta^i_\mu p_i\right)\rho^{(BA)}_{\nu}(t,t';\boldsymbol{p}) + I^{(\mathrm{g})}_{(\rho)\mu\nu}(t,t';\boldsymbol{p})\,, \tag{A.7a}$$
$$\left(\frac{\partial^2}{\partial t^2} + \boldsymbol{p}^2\right)\rho^{(AB)}_{\mu}(t,t';\boldsymbol{p}) = 0\,, \tag{A.7b}$$
$$\left(\frac{\partial^2}{\partial t^2} + \boldsymbol{p}^2\right)\rho^{(BA)}_{\mu}(t,t';\boldsymbol{p}) = 0\,, \tag{A.7c}$$
$$\left(\frac{\partial^2}{\partial t^2} + \boldsymbol{p}^2\right)\rho^{(BB)}(t,t';\boldsymbol{p}) = 0\,. \tag{A.7d}$$

Since the EoMs of the correlators involving the auxiliary field, and in particular of the $BA$-correlator, are free, we can solve their EoMs exactly by using their initial conditions (3.83), (3.82) and (3.84). We obtain:

$$\rho^{(AB)}_{\mu}(t,t';\boldsymbol{p}) = -\delta^0_\mu\cos(|\boldsymbol{p}|(t-t')) + \mathrm{i}\,\delta^i_\mu\frac{p_i}{|\boldsymbol{p}|}\sin(|\boldsymbol{p}|(t-t'))\,, \tag{A.8a}$$
$$\rho^{(BA)}_{\mu}(t,t';\boldsymbol{p}) = \delta^0_\mu\cos(|\boldsymbol{p}|(t-t')) - \mathrm{i}\,\delta^i_\mu\frac{p_i}{|\boldsymbol{p}|}\sin(|\boldsymbol{p}|(t-t'))\,, \tag{A.8b}$$
$$\rho^{(BB)}(t,t';\boldsymbol{p}) = 0\,. \tag{A.8c}$$

Plugging Eq. (A.8b) into the photon EoM (4.1a), we obtain:

$$\begin{aligned}\left(\frac{\partial^2}{\partial t^2} + \boldsymbol{p}^2\right)\rho^{(\mathrm{g})}_{\mu\nu}(t,t';\boldsymbol{p}) = -(1-\xi)|\boldsymbol{p}|\Bigg[&\left(\delta^0_\mu\delta^0_\nu + \delta^i_\mu\delta^j_\nu\frac{p_ip_j}{\boldsymbol{p}^2}\right)\sin(|\boldsymbol{p}|(t-t'))\\ &+\mathrm{i}(\delta^0_\mu\delta^i_\nu + \delta^i_\mu\delta^0_\nu)\frac{p_i}{|\boldsymbol{p}|}\cos(|\boldsymbol{p}|(t-t'))\Bigg] + I^{(\mathrm{g})}_{(\rho)}(t,t';\boldsymbol{p})\,.\end{aligned} \tag{A.9}$$

Of course, for vanishing memory integral, this equation reduces to free one (4.3).

Note that this whole derivation was only possible under the assumption that $\partial_\mu J^\mu = 0$. This is essentially the Ward identity in operator form. Therefore, it is only true for the *full* theory, not for a finitely truncated one.

[5] Note that the memory integral depends explicitly on the photon statistical and spectral functions, i. e. one actually has $I^{(\mathrm{g})}_{(\rho)}(F^{(\mathrm{g})},\rho^{(\mathrm{g})};x,y)$.

# Appendix B

# Gauge Invariant Quantities

The Ward identities are a manifestation of the gauge invariance of the theory, and in Sec. 5.2 it was shown that in a truncated theory they usually do not apply to correlation functions derived from the 2PI effective action. In particular, the photon self-energy obtained from a variation of the 2PI effective action with respect to the photon propagator (and which enters our EOMs) is *not* constrained to be transverse by the Ward identities.

In a very similar fashion, quantities which are gauge invariant in the exact theory will in general not be gauge invariant if derived from a truncation of the 2PI effective action. It follows that if quantities derived from the 1PI effective action which are known to be gauge invariant, this is usually not the case for the same quantities derived from the 2PI effective action since diagrams which are needed to cancel potential gauge dependencies are missing in its expansion to a given perturbative order.

It is also clear from the comparison of the diagrammatic expansion, however, that the gauge dependent terms are always of higher order in the coupling than the truncation of the 2PI effective action. In the case of a two-loop truncation, the perturbative expansions of the 1PI and 2PI effective actions agree up to the order of the truncation, and the diagram missing in the expansion of the 2PI effective action is obviously of order $\mathcal{O}(e^4)$.

As an illustration, let us consider some quantity $f$ which is gauge invariant if derived from the exact effective action, and denote by $\tilde{f}(\xi)$ the same quantity if derived from a two-loop truncated 2PI effective action.[1] According to what we have said above, it is then expected to have an expansion of the form

$$\tilde{f}(\xi) = \tilde{f}_0 + \tilde{f}_1 e^2 + \tilde{f}_2(\xi)\, e^4 + \mathcal{O}(e^6)\,, \tag{B.1}$$

i. e. the coefficient of the $\mathcal{O}(e^4)$-term will in general depend on the gauge fixing parameter.[2] It follows that the difference of the quantity for two values $\xi_1$ and $\xi_2$ of the gauge fixing

[1] We suppress all dependencies of the quantity (like on time, spatial momentum etc.) except for its dependence on the gauge fixing parameter.

[2] Especially for a weekly coupled theory like QED, the seeming smallness of the dependence on the gauge fixing parameter is deceptive. First, it means that for a sufficiently bad choice of the gauge parameter, the approximation of a quantity to its physical value can become arbitrarily bad, and it is not a priori clear what a good choice for the gauge parameter is since the exact dependence on it is unknown (see, however, Ref. [AS02]). And second, in real-time formulations derived from the 2PI effective action, a

parameter is given by

$$\tilde{f}(\xi_1) - \tilde{f}(\xi_2) = \Big[\tilde{f}_2(\xi_1) - \tilde{f}_2(\xi_2)\Big]\, e^4 + \mathcal{O}(e^6)\,. \tag{B.2}$$

It is therefore convenient to consider the difference of a given quantity for two gauge fixing parameters since then the parts which do not depend on the gauge fixing parameter cancel.

## B.1 The Photon Self-Energy

As an example, let us again consider the one-loop photon self-energy, Eq. (6.4). The exact one-loop 2PI photon self-energy is clearly gauge invariant:

$$\begin{aligned}
\Pi^{\mu\nu}(x,y) = \quad & e^2\,\mathrm{tr}(\gamma^\mu S(x,y)\gamma^\nu S(y,x)) \\
\mapsto \quad & e^2\,\mathrm{tr}(\gamma^\mu\, \mathrm{e}^{\mathrm{i}\,\Lambda(x)}\, S(x,y)\, \mathrm{e}^{-\,\mathrm{i}\,\Lambda(y)}\, \gamma^\nu\, \mathrm{e}^{\mathrm{i}\,\Lambda(y)}\, S(y,x)\, \mathrm{e}^{-\,\mathrm{i}\,\Lambda(x)}) \\
= \; & e^2\,\mathrm{tr}(\gamma^\mu S(x,y)\gamma^\nu S(y,x)) \\
= \; & \Pi^{\mu\nu}(x,y)\,.
\end{aligned}$$

Now consider its perturbative expansion when derived from the 1PI and 2PI effective action, respectively, which are diagrammatically shown in Fig. (5.2).

Just as for the transversality of the photon self-energy at each perturbative order, we have from the gauge invariance in each perturbative order:[3]

$$\frac{\partial}{\partial\xi}\Big[\Pi^{(2,1)\mu\nu}(x,y) + \Pi^{(2,2)\mu\nu}(x,y)\Big] = 0\,, \tag{B.3}$$

i. e. the dependence of the photon self-energy derived from the two-loop 2PI effective action on the gauge fixing parameter is given by

$$\begin{aligned}
&\frac{\partial}{\partial\xi}\,\Pi^{(2,1)\mu\nu}(x,y) \\
&\quad = -\frac{\partial}{\partial\xi}\,\Pi^{(2,2)\mu\nu}(x,y) + \mathcal{O}(e^6) \\
&\quad = -e^4 \int_{z,w} \mathrm{tr}\Big(\gamma^\mu S_0(x,z)\gamma^\rho S_0(z,y)\gamma^\nu S_0(y,w)\gamma^\sigma S_0(w,x)\Big)\frac{\partial}{\partial\xi} D_{0\rho\sigma}(z,w) + \mathcal{O}(e^6)\,.
\end{aligned} \tag{B.4}$$

coupling expansion is in fact an expansion in the product of coupling *and time*, so that a quantity which "superficially" is $\mathcal{O}(e^4)$ is in fact $\mathcal{O}(e^4\, m\, t)$ where $t$ denotes time and $m$ is some mass scale (like the fermion mass). This means that the dependence on the gauge parameter increases with time.

[3] Note that, contrary to the example (B.1) given in the previous section, in general we do not write the gauge fixing parameter explicitly as an argument.

## B.2 The Chiral Condensate

Another gauge invariant quantity is given by

$$S(x,x) = \int_{\boldsymbol{p}} F^{(\mathrm{f})}(x^0, x^0; \boldsymbol{p})\,. \tag{B.5}$$

That this quantity is gauge invariant immediately follows from the gauge transformation (2.8) of the fermion. In fact, the corresponding Lorentz components are separately gauge invariant. Since we consider only situations with vanishing macroscopic current densities, we have from the EOM (A.3a):

$$0 = \langle \partial_\mu F^{\mu\nu}(x)\rangle = \langle J^\nu(x)\rangle = 4e F_{\mathrm{V}}^{(\mathrm{f})\,\nu}(x,x) = 4e \int_{\boldsymbol{p}} F_{\mathrm{V}}^{(\mathrm{f})\,\nu}(x^0, x^0; \boldsymbol{p})\,,$$

i. e. the vector component vanishes identically. In fact, the spatial vector component also has to vanish due to isotropy, as is the case for the tensor component, so we are left with the scalar component

$$S_{\mathrm{S}}(x,x) = \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}p\, p^2\, F_{\mathrm{S}}^{(\mathrm{f})}(x^0, x^0; p)\,, \tag{B.6}$$

which is essentially the chiral condensate since $S_{\mathrm{S}}(x,x) = \langle \overline{\psi}(x)\psi(x)\rangle/4$.

## B.3 Two-Point Correlation Functions of the Electromagnetic Field Strength

Another gauge invariant quantity is the correlator of two electromagnetic field strength tensors (since each field strength is gauge invariant on its own), given by[4]

$$\begin{aligned}\mathcal{C}^{(FF)}_{\mu\nu\rho\sigma}(x,y) &= \langle F_{\mu\nu}(x) F_{\rho\sigma}(y)\rangle \\ &= \partial_{x\mu}\partial_{y\rho} D_{\nu\sigma}(x,y) - \partial_{x\mu}\partial_{y\sigma} D_{\nu\rho}(x,y) - \partial_{x\nu}\partial_{y\rho} D_{\mu\sigma}(x,y) + \partial_{x\nu}\partial_{y\sigma} D_{\mu\rho}(x,y)\,.\end{aligned} \tag{B.7}$$

In order to obtain a rank two tensor which is easier to handle, we contract the middle two indices to obtain:

$$\begin{aligned}\mathcal{C}^{(FF)}_{\mu\nu}(x,y) &:= \mathcal{C}^{(FF)}{}_{\mu}{}^{\rho}{}_{\rho\nu}(x,y) \\ &= \langle F_\mu^{\ \rho}(x) F_{\rho\nu}(y)\rangle \\ &= \partial_{x\mu}\partial_y^\rho D_{\rho\nu}(x,y) - \partial_{x\mu}\partial_{y\nu} D^\rho{}_\rho(x,y) - \partial_{x\rho}\partial_y^\rho D_{\mu\nu}(x,y) + \partial_x^\rho \partial_{y\nu} D_{\mu\rho}(x,y)\,.\end{aligned} \tag{B.8}$$

[4]Note that only in abelian gauge theories like QED the correlator can be expressed solely in terms of propagators; for non-abelian gauge theories, it would depend on the correlator of three and four gauge boson fields as well due to their self-interaction.

Performing a partial Fourier transformation with respect to space and decomposing it into its isotropic components, we obtain:

$$\begin{aligned}
\mathcal{C}^{(FF)}_{\mathrm{S}}(t,t';p) &= -\frac{\partial}{\partial t}\frac{\partial}{\partial t'}\Big[2D_{\mathrm{T}}(t,t';p)+D_{\mathrm{L}}(t,t';p)\Big] + p\left[\frac{\partial}{\partial t}\widetilde{D}_{\mathrm{V}_1}(t,t';p) - \frac{\partial}{\partial t'}\widetilde{D}_{\mathrm{V}_2}(t,t';p)\right] \\
&\quad + p^2 D_{\mathrm{S}}(t,t';p)\,, && \text{(B.9a)} \\
\widetilde{\mathcal{C}}^{(FF)}_{\mathrm{V}_1}(t,t';p) &= \phantom{-}2p\frac{\partial}{\partial t'}D_{\mathrm{T}}(t,t';p)\,, && \text{(B.9b)} \\
\widetilde{\mathcal{C}}^{(FF)}_{\mathrm{V}_2}(t,t';p) &= -2p\frac{\partial}{\partial t}D_{\mathrm{T}}(t,t';p)\,, && \text{(B.9c)} \\
\mathcal{C}^{(FF)}_{\mathrm{T}}(t,t';p) &= -\left(\frac{\partial}{\partial t}\frac{\partial}{\partial t'} - p^2\right)D_{\mathrm{T}}(t,t';p)\,, && \text{(B.9d)} \\
\mathcal{C}^{(FF)}_{\mathrm{L}}(t,t';p) &= -\frac{\partial}{\partial t}\frac{\partial}{\partial t'}D_{\mathrm{L}}(t,t';p) + p\left[\frac{\partial}{\partial t}\widetilde{D}_{\mathrm{V}_1}(t,t';p) - \frac{\partial}{\partial t'}\widetilde{D}_{\mathrm{V}_2}(t,t';p)\right] \\
&\quad + p^2\Big[D_{\mathrm{S}}(t,t';p) - 2D_{\mathrm{T}}(t,t';p)\Big]\,. && \text{(B.9e)}
\end{aligned}$$

The isotropic components can then be decomposed into their statistical and spectral parts, and for a free theory, they are given by:

$$\begin{aligned}
\mathcal{C}^{(FF)}_{0(\rho)\mathrm{S}}(t,t';p) &= \phantom{-}2p\sin(p\,(t-t'))\,, && \text{(B.10a)} \\
\widetilde{\mathcal{C}}^{(FF)}_{0(\rho)\mathrm{V}_1}(t,t';p) = \widetilde{\mathcal{C}}^{(FF)}_{0(\rho)\mathrm{V}_2}(t,t';p) = \widetilde{\mathcal{C}}^{(FF)}_{0(\rho)\mathrm{V}}(t,t';p) &= \phantom{-}2p\cos(p\,(t-t'))\,, && \text{(B.10b)} \\
\mathcal{C}^{(FF)}_{0(\rho)\mathrm{T}}(t,t';p) &= \phantom{-}0\,, && \text{(B.10c)} \\
\mathcal{C}^{(FF)}_{0(\rho)\mathrm{L}}(t,t';p) &= -2p\sin(p\,(t-t'))\,, && \text{(B.10d)}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{C}^{(FF)}_{0(\mathrm{F})\mathrm{S}}(t,t';p) &= \phantom{-}2p\left[\frac{1}{2}+n^{(\mathrm{g})}(p)\right]\cos(p\,(t-t'))\,, && \text{(B.11a)} \\
\widetilde{\mathcal{C}}^{(FF)}_{0(\mathrm{F})\mathrm{V}_1}(t,t';p) = \widetilde{\mathcal{C}}^{(FF)}_{0(\mathrm{F})\mathrm{V}_2}(t,t';p) = \widetilde{\mathcal{C}}^{(FF)}_{0(\mathrm{F})\mathrm{V}}(t,t';p) &= -2p\left[\frac{1}{2}+n^{(\mathrm{g})}(p)\right]\sin(p\,(t-t'))\,, && \text{(B.11b)} \\
\mathcal{C}^{(FF)}_{0(\mathrm{F})\mathrm{T}}(t,t';p) &= \phantom{-}0\,, && \text{(B.11c)} \\
\mathcal{C}^{(FF)}_{0(\mathrm{F})\mathrm{L}}(t,t';p) &= -2p\left[\frac{1}{2}+n^{(\mathrm{g})}(p)\right]\cos(p\,(t-t'))\,. && \text{(B.11d)}
\end{aligned}$$

Obviously, the free solutions do not depend on the gauge fixing parameter, as expected.

## B.4 The Energy-Momentum Tensor

### B.4.1 Definition

There are essentially two ways to obtain the energy-momentum tensor of a field theory: As the Noether current corresponding to the invariance of the theory under time translations (the *canonical* energy-momentum tensor), or as the variation of the (matter) action with respect to the metric. For a quantum field theory, in the first case one obtains an operator expression, and one has to calculate its expectation value. In the second case, on the other hand, it turns out that all one has to do is to replace the classical action with the effective action. Therefore, this definition is much better suited for our purposes, since it allows us to derive an expression for the energy momentum tensor which depends on the one- and two-point correlation functions which are solutions to the 2PI EOMs.

In general, the energy-momentum tensor is then given by:

$$T^{\mu\nu}(x) = \frac{2}{\sqrt{-g(x)}} \left.\frac{\delta\Gamma[\Phi, g]}{\delta g_{\mu\nu}(x)}\right|_{g_{\mu\nu}=\overline{g}_{\mu\nu}} \tag{B.12}$$

(where $\Phi$ collectively denotes all correlation functions the effective action depends on[5]). Note that $g_{\mu\nu}$ is *not* the physical metric, but a variational parameter. The final expression which is obtained by varying the effective action then has to be evaluated for the physical metric $\overline{g}_{\mu\nu}$, which in our case will always be the Minkowski metric $\eta_{\mu\nu}$.[6] In this case, $\sqrt{-\eta} = 1$, so that the definition of the energy-momentum tensor in Minkowski spacetime can be simplified to:

$$T^{\mu\nu}(x) = 2 \left.\frac{\delta\Gamma[\Phi, g]}{\delta g_{\mu\nu}(x)}\right|_{g_{\mu\nu}=\eta_{\mu\nu}} . \tag{B.13}$$

One important advantage of employing this definition of the energy-momentum tensor (in contrast to the canonical energy-momentum tensor) is that it is gauge invariant and symmetric in the first place (and manifestly so).

Since for the full theory, the 1PI effective action and the 2PI effective action are equal, we will just replace the 1PI effective action in the definition of the energy-momentum tensor by the 2PI effective action in order to obtain an expression which depends on propagators only and includes all quantum fluctuations. The convenience of using this definition instead of the canonical one becomes obvious as soon as we consider a truncated theory: With our definition, we still only work with known quantities (i. e. the propagators), while for the canonical energy-momentum tensor, one has to calculate expectation values of quantum field operators which do not easily translate to propagators for a truncated theory. Another nontrivial problem is the fact that, since the energy-momentum tensor is a one-point function, one has to evaluate quantum field operators (or correlation functions after taking expectation values[7]) at equal spacetime points, which requires some form of regularization [BD82, Chr76] in order to make it well-defined.

[5] I. e. the field expectation values of all fields for the standard (1PI) effective action, the field expectation values and connected propagators of all fields for the 2PI effective actions, etc.

[6] Only in this section, $g_{\mu\nu}$ denotes an arbitrary metric, while we denote the Minkowski metric by $\eta_{\mu\nu}$.

[7] This is not a problem if the energy-momentum tensor is defined with respect to the 1PI effective action since it only depends on one-point functions itself. If it is defined with respect to $n$PI effective actions (for $n > 1$), however, it is a problem.

### B.4.2 Properties

The energy-momentum tensor is a symmetric second-rank tensor, i. e. it depends on 10 functions in general: The energy density $T^{00}(x)$, the energy flux or momentum density $T^{i0}(x) = T^{0i}(x)$, and the momentum flux $T^{ij}(x) = T^{ji}(x)$. For $i = j$, this is the normal stress, while it is the shear stress for $i \neq j$.[8]

A (with respect to spacetime) homogeneous energy-momentum tensor is always constant (this is true for every homogeneous one-point function). Correspondingly, a spatially homogeneous energy-momentum tensor (or one-point function in general) can only depend on time.

In vacuum, there is only one symmetric second-rank tensor which can be used as a basis to represent the energy-momentum tensor, which is the metric. Since the vacuum is spacetime homogeneous, its energy-momentum tensor is constant, i. e. $T^{\mu\nu} = \Lambda g^{\mu\nu}$. $\Lambda$ resembles the cosmological constant, and it is equal to the (vacuum) energy density.

In an isotropic system, there are two symmetric second-rank tensors which can be used as a basis for the energy-momentum tensor: The metric and the tensor product of the four-velocity of the system $n^\mu n^\nu$. We therefore have $T^{\mu\nu}(x) = \alpha(x) n^\mu n^\nu + \beta(x) g^{\mu\nu} = \Big[\varepsilon(x) + p(x)\Big] n^\mu n^\nu - p(x) g^{\mu\nu} = \varepsilon(x) n^\mu n^\nu - (g^{\mu\nu} - n^\mu n^\nu) p(x)$, i. e. energy density is the projection of the energy-momentum tensor along the direction of the four-velocity of the system, while (negative) pressure is the projection onto the hypersurface perpendicular to it:

$$\varepsilon(x) = n_\mu n_\nu T^{\mu\nu}(x)\,, \tag{B.14}$$

$$p(x) = -\frac{1}{3}\,(g_{\mu\nu} - n_\mu n_\nu) T^{\mu\nu}(x) = -\frac{1}{3}\Big[T^\mu{}_\mu(x) - \varepsilon(x)\Big] = -\frac{1}{3}\,T^i{}_i(x)\,. \tag{B.15}$$

A homogeneous, isotropic system is hence described by two (SO(3)-)scalar functions of time only, and they are related by the *equation of state* (EoS) of the system. A dimensionless quantity characterizing the system is then given by the ratio of pressure and energy density, $w(t) = p(t)/\varepsilon(t)$.

### B.4.3 Energy-Momentum Tensor from the 2PI Effective Action

Since the energy-momentum tensor is a thermodynamic quantity, we have to include Faddeev-Popov ghosts in order to obtain the right result.[9] We will therefore consider the contributions to the energy density which are due to fermions only, photons only, ghosts only, and the interaction between fermions and photons separately.[10] Therefore, we

[8] In an isotropic system, there is no shear stress, so every off-diagonal element vanishes identically.

[9] In calculating correlation functions, this is not necessary in linearly gauge-fixed abelian gauge theories since the ghosts decouple from the rest and therefore cancel due to the normalization. It is, however, intuitively clear that ghosts carry energy and momentum and therefore contribute to the energy-momentum tensor.

[10] By this we mean that the corresponding parts of the effective action contain only fermion propagators, only photon propagators, or both. Since propagators are always resummed, it is of course not quite correct

write the effective action as

$$\Gamma[D,S,G] = \Gamma^{(\mathrm{g})}[D] + \Gamma^{(\mathrm{f})}[S] + \Gamma^{(\mathrm{gh})}[G] + \Gamma_{2\mathrm{PI}}[D,S] \tag{B.16}$$

with the photon part

$$\Gamma^{(\mathrm{g})}[D] = \frac{\mathrm{i}}{2}\,\mathrm{Tr}\ln\Big(D^{-1}\Big) + \frac{\mathrm{i}}{2}\,\mathrm{Tr}\Big(D_0^{-1}(D-D_0)\Big)\,, \tag{B.17}$$

the fermion part

$$\Gamma^{(\mathrm{f})}[D] = -\,\mathrm{i}\,\mathrm{Tr}\ln\Big(S^{-1}\Big) - \mathrm{i}\,\mathrm{Tr}\Big(S_0^{-1}(S-S_0)\Big)\,, \tag{B.18}$$

and the ghost part

$$\Gamma^{(\mathrm{gh})}[G] = -\,\mathrm{i}\,\mathrm{Tr}\ln\Big(G^{-1}\Big) - \mathrm{i}\,\mathrm{Tr}\Big(G_0^{-1}(G-G_0)\Big)\,. \tag{B.19}$$

Note that the respective first terms in the photon and fermion part do not contribute to the energy-momentum tensor since they are independent of the metric. The only interesting terms aside from the 2PI part of the effective action are hence given by $\mathrm{i}\,\mathrm{Tr}\Big(D_0^{-1}D\Big)/2$, $-\,\mathrm{i}\,\mathrm{Tr}\Big(S_0^{-1}S\Big)$ and $-\,\mathrm{i}\,\mathrm{Tr}\Big(G_0^{-1}G\Big)$.

**Useful Relations** Before doing the actual calculation, we start by collecting some useful relations concerning variations with respect to the metric:

For the variation of a metric with respect to itself, we obtain:[11]

$$\begin{aligned}
\frac{\delta g^{\rho\sigma}(y)}{\delta g^{\mu\nu}(x)} &= \frac{1}{2}\Big(\delta^\rho_\mu\delta^\sigma_\nu + \delta^\sigma_\mu\delta^\rho_\nu\Big)\delta^4(y-x)\,,\\
\frac{\delta g_{\rho\sigma}(y)}{\delta g^{\mu\nu}(x)} &= -\frac{1}{2}\Big[g_{\mu\rho}(y)g_{\nu\sigma}(y) + g_{\mu\sigma}(y)g_{\nu\rho}(y)\Big]\delta^4(y-x)\,.
\end{aligned}$$

Since the effective action has to be varied with respect to the metric, one has to be careful not to overlook places in which the metric appears. For instance, there is a metric involved in each inner product of two contravariant or two covariant quantities (while

to speak of *purely* fermionic or *purely* photonic contributions. In fact, what we call "purely fermionic" or "purely photonic" contributions are contributions from one-loop diagrams (with resummed propagators), while the interaction contribution comes from the 2PI contribution to the effective action, i. e. from everything beyond one-loop.

[11]The following identities can be obtained by writing the metric $g = (g_{\mu\nu})$ in matrix form (with inverse metric $g^{-1} = (g^{\mu\nu})$) and incorporating the fact that the result of a variation with respect to the metric must be symmetric in the same indices as the metric by symmetrization:

$$0 = \delta\mathbf{1} = \delta\big(g^{-1}g\big) = \big(\delta g^{-1}\big)g + g^{-1}\delta g\,,$$

so that

$$\delta g^{-1} = -g^{-1}(\delta g)g^{-1}\,,$$

or componentwise:

$$\delta g^{\mu\nu} = -g^{\mu\rho}\delta g_{\rho\sigma}g^{\sigma\nu}\,.$$

there is none involved in the inner product of one contravariant and one covariant quantity). Further, since the variational metric is not constrained to be flat, we have to replace (spacetime) integral measures $\mathrm{d}^4x$ by $\sqrt{-g(x)}\,\mathrm{d}^4x$ where $-g(x)$ is the determinant of the metric (i. e. unity for the Minkowski metric, in which case the measure reduces to the usual one). Another place where the metric occurs are covariant derivatives (which have to be used instead of partial derivatives for a general spacetime). A variation of a covariant derivative with respect to the metric, however, always vanishes when evaluated for Minkowski spacetime so it is not necessary to replace partial with covariant derivatives for our concerns.

One also has to be careful not to introduce metrics where they do in fact *not* appear. It is therefore necessary to know what kind of tensors the appearing quantities *naturally* are. The gradient, for instance, is naturally covariant (i. e. a tensor of type $(0,1)$). In particular, the contraction of two co- or two contravariant indices requires a metric, so there is always a metric involved in the d'Alembertian, for instance. The gamma matrices, however, are naturally contravariant, so there is *no* metric involved in their contraction with the gradient, i. e.

$$\frac{\delta}{\delta g^{\mu\nu}(x)}\Big[\gamma^\rho(y)\partial_{y\rho}\Big] = 0\,.$$

The important point here is that a tensorial object which appears in a form which does not correspond to its natural type (e. g. a $(0,1)$ vector appears as a $(1,0)$ vector) is not independent of the metric, i. e., assuming that $v$ is a covariant vector appearing in a contravariant form:

$$\frac{\delta v^\rho(y)}{\delta g^{\mu\nu}(x)} = \frac{\delta\Big[g^{\rho\sigma}(y)v_\sigma(y)\Big]}{\delta g^{\mu\nu}(x)} = \frac{1}{2}\Big(\delta^\rho_\mu\delta^\sigma_\nu+\delta^\sigma_\mu\delta^\rho_\nu\Big)v_\sigma(y)\delta^4(y-x) = \frac{1}{2}\Big[\delta^\rho_\nu v_\mu(y)+\delta^\rho_\mu v_\nu(y)\Big]\delta^4(y-x)\,.$$

We can then distinguish four different cases for the variation of a contraction with respect to the metric, depending on the tensorial nature of the objects. We have:

- $v$ is a contravariant vector and $w$ is a contravariant vector:

$$\begin{aligned}\frac{\delta}{\delta g^{\mu\nu}(x)}\Big[v_\rho(y)w^\rho(y)\Big] &= \frac{\delta}{\delta g^{\mu\nu}(x)}\Big[g_{\rho\sigma}(y)v^\sigma(y)w^\rho(y)\Big]\\ &= -\frac{1}{2}\Big[g_{\mu\rho}(y)g_{\nu\sigma}(y) + g_{\mu\sigma}(y)g_{\nu\rho}(y)\Big]v^\sigma(y)w^\rho(y)\delta^4(y-x)\\ &= -\frac{1}{2}\Big[v_\mu(y)w_\nu(y) + v_\nu(y)w_\mu(y)\Big]\delta^4(y-x)\,,\end{aligned}$$

- $v$ is a covariant vector and $w$ is a covariant vector:

$$\begin{aligned}\frac{\delta}{\delta g^{\mu\nu}(x)}\Big[v_\rho(y)w^\rho(y)\Big] &= \frac{\delta}{\delta g^{\mu\nu}(x)}\Big[g^{\rho\sigma}(y)v_\sigma(y)w_\rho(y)\Big]\\ &= \frac{1}{2}\Big[\delta^\rho_\mu(y)\delta^\sigma_\nu(y) + \delta^\sigma_\mu(y)\delta^\rho_\nu(y)\Big]v_\sigma(y)w_\rho(y)\delta^4(y-x)\\ &= \frac{1}{2}\Big[v_\mu(y)w_\nu(y) + v_\nu(y)w_\mu(y)\Big]\delta^4(y-x)\,,\end{aligned}$$

- $v$ is a covariant vector and $w$ is a contravariant vector:

$$\frac{\delta}{\delta g^{\mu\nu}(x)}\Big[v_\rho(y)w^\rho(y)\Big] = 0\,,$$

- $v$ is a contravariant vector and $w$ is a covariant vector:

$$\begin{aligned}
&\frac{\delta}{\delta g^{\mu\nu}(x)}\Big[v_\rho(y)w^\rho(y)\Big] \\
&= \frac{\delta}{\delta g^{\mu\nu}(x)}\Big[g_{\rho\sigma}(y)g^{\rho\tau}(y)v^\sigma(y)w_\tau(y)\Big] \\
&= \bigg\{-\frac{1}{2}\Big[g_{\mu\rho}(y)g_{\nu\sigma}(y)+g_{\mu\sigma}(y)g_{\nu\rho}(y)\Big]g^{\rho\tau}(y)+\frac{1}{2}\,g_{\rho\sigma}(y)\Big(\delta^\rho_\mu\delta^\tau_\nu+\delta^\tau_\mu\delta^\rho_\nu\Big)\bigg\} \\
&\qquad \cdot v^\sigma(y)w_\tau(y)\delta^4(y-x) \\
&= \frac{1}{2}\bigg\{g_{\mu\sigma}(y)\delta^\tau_\nu+g_{\nu\sigma}(y)\delta^\tau_\mu-\Big[g_{\nu\sigma}(y)\delta^\tau_\mu+g_{\mu\sigma}(y)\delta^\tau_\nu\Big]\bigg\}\,v^\sigma(y)w_\tau(y)\delta^4(y-x) \\
&= 0\,.
\end{aligned}$$

If we denote a Lorentz tensor with $m$ contravariant and $n$ covariant indices as a tensor of type $(m,n)$, then the partial derivative $\partial_\mu$ is a type $(0,1)$ tensor, the gamma matrices $\gamma^\mu$ form a type $(1,0)$ tensor, the photon propagator is a type $(0,2)$ tensor, and the inverse photon propagator is a type $(2,0)$ tensor.

With these relations at hand, we can now calculate the separate pieces of the energy-momentum tensor.

### B.4.4 Ghost Part

We start with the ghost part, which is the easiest one to calculate. The free inverse ghost propagator is given by:

$$G_0^{-1}(x,y) = \mathrm{i}\,\Box_x\delta^4(x-y)\,. \tag{B.20}$$

We then have:

$$\begin{aligned}
-\,\mathrm{i}\,\mathrm{Tr}\Big(G_0^{-1}G\Big) &= -\,\mathrm{i}\int_{z,y}\sqrt{-g(y)}\,\sqrt{-g(z)}\,G_0^{-1}(y,z)G(z,y) \\
&= \int_{y,z}\sqrt{-g(y)}\,\sqrt{-g(z)}\,\Big[\Box_y\delta^4(y-z)\Big]G(z,y) \\
&= \int_{y,z}\sqrt{-g(y)}\,\sqrt{-g(z)}\,\Big[g^{\rho\sigma}(y)\partial_{y\rho}\partial_{y\sigma}\delta^4(y-z)\Big]G(z,y) \\
&= \int_{y,z}\sqrt{-g(y)}\,\sqrt{-g(z)}\,\delta^4(y-z)\,g^{\rho\sigma}(y)\partial_{y\rho}\partial_{y\sigma}G(z,y) \\
&= \int_z\sqrt{-g(z)}\,g^{\rho\sigma}(z)\partial_{z\rho}\partial_{z\sigma}F^{(\mathrm{gh})}(z,z)\,,
\end{aligned}$$

where we have integrated by parts[12], and in the last step we have used that $G(z,z) = F^{(\mathrm{gh})}(z,z)$. The propagator evaluated at equal spacetime points is not well-defined, and

[12]Note that in fact, there are additional terms containing derivatives of the metric. However, since in the end we will only be interested in evaluating the expressions for the Minkowski metric (which is constant), we have discarded those terms in the first place.

the corresponding term has to be regularized according to[13]

$$\partial_{z\rho}\partial_{z\sigma}F^{(\mathrm{gh})}(z,z) \to -\partial_{z\rho}\partial_{z'\sigma}F^{(\mathrm{gh})}(z,z')|_{z'=z}\,. \tag{B.21}$$

It follows that

$$\begin{aligned}
\frac{\delta\Gamma^{(\mathrm{gh})}[G]}{\delta g^{\mu\nu}(x)} &= -\frac{1}{2}\int_z \sqrt{-g(z)}\,\delta^4(z-x)\Big[-g_{\mu\nu}(z)g^{\rho\sigma}(z) + \Big(\delta^\rho_\mu\delta^\sigma_\nu + \delta^\rho_\nu\delta^\sigma_\mu\Big)\Big]\partial_{z'\rho}\partial_{z\sigma}F^{(\mathrm{gh})}(z,z')\Big|_{z'=z} \\
&= \frac{1}{2}\Big[g_{\mu\nu}(x)g^{\rho\sigma}(x) - \Big(\delta^\rho_\mu\delta^\sigma_\nu + \delta^\rho_\nu\delta^\sigma_\mu\Big)\Big]\partial_{x'\rho}\partial_{x\sigma}F^{(\mathrm{gh})}(x,x')\Big|_{x'=x}\,,
\end{aligned}$$

so that the ghost part of the energy-momentum tensor reads:

$$\begin{aligned}
T^{(\mathrm{gh})}_{\mu\nu}(x) &= 2\,\frac{\delta\Gamma^{(\mathrm{gh})}[G]}{\delta g^{\mu\nu}(x)}\Big|_{g_{\mu\nu}(x)=\eta_{\mu\nu}} \\
&= \Big[\eta_{\mu\nu}\eta^{\rho\sigma} - \Big(\delta^\rho_\mu\delta^\sigma_\nu + \delta^\rho_\nu\delta^\sigma_\mu\Big)\Big]\partial_{x'\rho}\partial_{x\sigma}F^{(\mathrm{gh})}(x,x')\Big|_{x'=x} \\
&= \Big[\eta_{\mu\nu}\partial_{x'\rho}\partial^\rho_x - \Big(\partial_{x'\mu}\partial_{x\nu} + \partial_{x'\nu}\partial_{x\mu}\Big)\Big]F^{(\mathrm{gh})}(x,x')\Big|_{x'=x} \\
&= \int_{\boldsymbol{p}}\Bigg\{\eta_{\mu\nu}\left(\frac{\partial}{\partial x^0}\frac{\partial}{\partial x'^0} - \boldsymbol{p}^2\right) \\
&\qquad - \Bigg[\quad\left(\delta^0_\mu\,\frac{\partial}{\partial x'^0} + \mathrm{i}\,\delta^i_\mu p_i\right)\left(\delta^0_\nu\,\frac{\partial}{\partial x^0} - \mathrm{i}\,\delta^j_\nu p_j\right) \\
&\qquad\quad + \left(\delta^0_\nu\,\frac{\partial}{\partial x'^0} + \mathrm{i}\,\delta^i_\nu p_i\right)\left(\delta^0_\mu\,\frac{\partial}{\partial x^0} - \mathrm{i}\,\delta^j_\mu p_j\right)\Bigg]\Bigg\}\,F^{(\mathrm{gh})}\Big(x^0,x'^0;\boldsymbol{p}\Big)\Big|_{x'^0=x^0} \\
&= -\int_{\boldsymbol{p}}\Bigg[\left(\delta^0_\mu\delta^0_\nu + \delta^i_\mu\delta^j_\nu\,\frac{p_ip_j}{\boldsymbol{p}^2}\right)\left(\frac{\partial}{\partial x^0}\frac{\partial}{\partial x'^0} + \boldsymbol{p}^2\right) \\
&\qquad - \mathrm{i}\Big(\delta^0_\mu\delta^i_\nu + \delta^0_\nu\delta^i_\mu\Big)\,p_i\left(\frac{\partial}{\partial x^0} - \frac{\partial}{\partial x'^0}\right)\Bigg]F^{(\mathrm{gh})}\Big(x^0,x'^0;\boldsymbol{p}\Big)\Big|_{x'^0=x^0} \\
&= -2\int_{\boldsymbol{p}}|\boldsymbol{p}|\left[\frac{1}{2} + n^{(\mathrm{g})}(\boldsymbol{p})\right]\left(\delta^0_\mu\delta^0_\nu + \delta^i_\mu\delta^j_\nu\,\frac{p_ip_j}{\boldsymbol{p}^2}\right)\,,
\end{aligned} \tag{B.22}$$

where in the last line we have used that since the ghosts do not couple to anything, their propagator is that of a free massless scalar field,

$$F^{(\mathrm{gh})}\Big(x^0,x'^0;\boldsymbol{p}\Big) = \left[\frac{1}{2} + n^{(\mathrm{g})}(\boldsymbol{p})\right]\frac{\cos(|\boldsymbol{p}|(x^0-x'^0))}{|\boldsymbol{p}|}\,. \tag{B.23}$$

[13]To be more precise, we have

$$\begin{aligned}
\partial_{z\rho}\partial_{z\sigma}F^{(\mathrm{gh})}(z,z) \to\quad & -\frac{1}{2}\Big[\partial_{z\rho}\partial_{z'\sigma}F^{(\mathrm{gh})}(z,z') + \partial_{z'\rho}\partial_{x\sigma}F^{(\mathrm{gh})}(z',z)\Big]\Big|_{z'=z} \\
&= -\frac{1}{2}\left(\partial_{z\rho}\partial_{z'\sigma} + \partial_{z'\rho}\partial_{z\sigma}\right)F^{(\mathrm{gh})}(z,z')\big|_{z'=z}\,.
\end{aligned}$$

However, since this term only appears in contracted form in the action, we effectively have (B.21).

Note that one could have guessed this result exactly: The minus sign is due to the fermionic (anti-)commutation relations; the factor of 2 is due to the two ghosts; the factor of $|\boldsymbol{p}|$ is clear for dimensional reasons (we need to have dimensions of energy-momentum); the term involving the distribution function is due to the bosonic statistics of ghosts; and the tensor structure is clear from the fact that the pressure density of a free massless gas is minus one third of its energy density.

### B.4.5 Photon Part

We have:

$$
\begin{aligned}
&\frac{\mathrm{i}}{2}\,\mathrm{Tr}\Big(D_0^{-1}D\Big)\\
&\quad= \frac{\mathrm{i}}{2}\int_{y,z}\sqrt{-g(y)}\,\sqrt{-g(z)}\,\Big(D_0^{-1}\Big)^{\rho\sigma}(y,z)D_{\sigma\rho}(z,y)\\
&\quad= \frac{1}{2}\int_{y,z}\sqrt{-g(y)}\,\sqrt{-g(z)}\,\bigg[g^{\rho\sigma}(y)g^{\alpha\beta}(y)-\bigg(1-\frac{1}{\xi}\bigg)\,g^{\rho\alpha}(y)g^{\sigma\beta}(y)\bigg]\Big[\partial_{y\alpha}\partial_{y\beta}\,\delta^4(y-z)\Big]D_{\sigma\rho}(z,y)\\
&\quad= \frac{1}{2}\int_{y,z}\sqrt{-g(y)}\,\sqrt{-g(z)}\,\delta^4(y-z)\,\bigg[g^{\rho\sigma}(y)g^{\alpha\beta}(y)-\bigg(1-\frac{1}{\xi}\bigg)\,g^{\rho\alpha}(y)g^{\sigma\beta}(y)\bigg]\,\partial_{y\alpha}\partial_{y\beta}\,D_{\sigma\rho}(z,y)\\
&\quad= \frac{1}{2}\int_{z}\sqrt{-g(z)}\,\bigg[g^{\rho\sigma}(z)g^{\alpha\beta}(z)-\bigg(1-\frac{1}{\xi}\bigg)\,g^{\rho\alpha}(z)g^{\sigma\beta}(z)\bigg]\,\partial_{z\alpha}\partial_{z\beta}F^{(\mathrm{g})}_{\sigma\rho}(z,z)
\end{aligned}
$$

where we have integrated by parts in order to get rid of the derivatives of the delta distribution[14], and in the last step we have used that $D_{\sigma\rho}(z,z)=F^{(\mathrm{g})}_{\sigma\rho}(z,z)$. The question how to regularize this expression is more complicated than for the ghosts, since here we have uncontracted Lorentz indices and hence in a sense less symmetry. The correct regularization can be found by considering the classical effective action (i. e. the classical action including the gauge fixing term) in terms of quantum field operators. The gauge invariant part has to be regularized as

$$
\begin{aligned}
-\frac{1}{4}\,\widehat{F}_{\mu\nu}(x)\widehat{F}^{\mu\nu}(x) &\to -\frac{1}{2}\left[\widehat{F}_{\mu\nu}(x)\widehat{F}^{\mu\nu}(x')+\widehat{F}_{\mu\nu}(x')F^{\mu\nu}(x)\right]\bigg|_{x'=x}\\
&\to -\frac{1}{2}\left[\partial_{x\mu}\partial^{\mu}_{x'}F^{(\mathrm{g})\nu}{}_{\nu}(x,x')-\partial^{\mu}_{x}\partial^{\nu}_{x'}F^{(\mathrm{g})}_{\nu\mu}(x,x')\right]\bigg|_{x'=x}
\end{aligned}
\tag{B.24}
$$

(where in the second line we have discarded total derivatives which vanish in the action), while the gauge fixing part has to be regularized according to [BCER08]

$$
\begin{aligned}
-\frac{1}{2\xi}\left[\partial^{\mu}\widehat{A}_{\mu}(x)\right]\left[\partial^{\nu}\widehat{A}_{\nu}(x)\right] &\to -\frac{1}{4\xi}\Big\{\left[\partial^{\mu}_{x}\widehat{A}_{\mu}(x)\right]\left[\partial^{\nu}_{x'}\widehat{A}_{\nu}(x')\right]+\left[\partial^{\mu}_{x'}\widehat{A}_{\mu}(x')\right]\left[\partial^{\nu}_{x}\widehat{A}_{\nu}(x)\right]\Big\}\bigg|_{x'=x}\\
&\to -\frac{1}{2\xi}\,\partial^{\mu}_{x}\partial^{\nu}_{x'}F^{(\mathrm{g})}_{\mu\nu}(x,x')\bigg|_{x'=x}\ .
\end{aligned}
\tag{B.25}
$$

[14]Note that we ignore derivatives of the metric since they vanish at the end when evaluated for the Minkowski metric.

Note that without regularization, the second term of the gauge invariant part and the gauge fixing part would combine to form the term

$$\frac{1}{2}\left(1-\frac{1}{\xi}\right)\partial_x^\mu\partial_x^\nu F^{(\mathrm{g})}_{\nu\mu}(x,x)\,;$$

however, due to the regularization, we in fact obtain

$$\frac{1}{2}\,\partial_x^\mu\partial_{x'}^\nu\left[F^{(\mathrm{g})}_{\nu\mu}(x,x')-\frac{1}{\xi}\,F^{(\mathrm{g})}_{\mu\nu}(x,x')\right]\bigg|_{x'=x}\,. \tag{B.26}$$

With this regularization, we then have:

$$\begin{aligned}
&\frac{\mathrm{i}}{2}\,\mathrm{Tr}\Big(D_0^{-1}D\Big)\\
&\quad=-\frac{1}{2}\int_z\sqrt{-g(z)}\left\{g^{\rho\sigma}(z)g^{\alpha\beta}(z)-\left[g^{\rho\alpha}(z)g^{\sigma\beta}(z)-\frac{1}{\xi}\,g^{\rho\beta}(z)g^{\sigma\alpha}(z)\right]\right\}\partial_{z\alpha}\partial_{z'\beta}F^{(\mathrm{g})}_{\sigma\rho}(z,z')\bigg|_{z'=z}\,.
\end{aligned}$$

It then follows that:

$$\begin{aligned}
&\frac{\delta\Gamma^{(\mathrm{g})}[D,g]}{\delta g^{\mu\nu}(x)}\\
&\quad=-\frac{1}{4}\int_z\sqrt{-g(z)}\,\delta^4(z-x)\\
&\qquad\quad\cdot\left\{-g_{\mu\nu}(z)\left[g^{\rho\sigma}(z)g^{\alpha\beta}(z)-g^{\rho\alpha}(z)g^{\sigma\beta}(z)+\frac{1}{\xi}\,g^{\rho\beta}(z)g^{\sigma\alpha}(z)\right]\right.\\
&\qquad\qquad+\Big(\delta^\rho_\mu\delta^\sigma_\nu+\delta^\rho_\nu\delta^\sigma_\mu\Big)g^{\alpha\beta}(z)+g^{\rho\sigma}(z)\Big(\delta^\alpha_\mu\delta^\beta_\nu+\delta^\alpha_\nu\delta^\beta_\mu\Big)\\
&\qquad\qquad-\left[\Big(\delta^\rho_\mu\delta^\alpha_\nu+\delta^\rho_\nu\delta^\alpha_\mu\Big)g^{\sigma\beta}(z)+g^{\rho\alpha}(z)\Big(\delta^\sigma_\mu\delta^\beta_\nu+\delta^\sigma_\nu\delta^\beta_\mu\Big)\right]\\
&\qquad\qquad\left.+\frac{1}{\xi}\left[\Big(\delta^\rho_\mu\delta^\beta_\nu+\delta^\rho_\nu\delta^\beta_\mu\Big)g^{\sigma\alpha}(z)+g^{\rho\beta}(z)\Big(\delta^\sigma_\mu\delta^\alpha_\nu+\delta^\sigma_\nu\delta^\alpha_\mu\Big)\right]\right\}\partial_{z\alpha}\partial_{z'\beta}F^{(\mathrm{g})}_{\sigma\rho}(z,z')\bigg|_{z'=z}\\
&\quad=-\frac{1}{4}\left\{-g_{\mu\nu}(x)\left[g^{\rho\sigma}(x)g^{\alpha\beta}(x)-g^{\rho\alpha}(x)g^{\sigma\beta}(x)+\frac{1}{\xi}\,g^{\rho\beta}(x)g^{\sigma\alpha}(x)\right]\right.\\
&\qquad\quad+\Big(\delta^\rho_\mu\delta^\sigma_\nu+\delta^\rho_\nu\delta^\sigma_\mu\Big)g^{\alpha\beta}(x)+g^{\rho\sigma}(x)\Big(\delta^\alpha_\mu\delta^\beta_\nu+\delta^\alpha_\nu\delta^\beta_\mu\Big)\\
&\qquad\quad-\Big[\Big(\delta^\rho_\mu\delta^\alpha_\nu+\delta^\rho_\nu\delta^\alpha_\mu\Big)g^{\sigma\beta}(x)+g^{\rho\alpha}(x)\Big(\delta^\sigma_\mu\delta^\beta_\nu+\delta^\sigma_\nu\delta^\beta_\mu\Big)\\
&\qquad\quad\left.+\frac{1}{\xi}\Big[\Big(\delta^\rho_\mu\delta^\beta_\nu+\delta^\rho_\nu\delta^\beta_\mu\Big)\Big]g^{\sigma\alpha}(x)+g^{\rho\beta}(x)\Big(\delta^\sigma_\mu\delta^\alpha_\nu+\delta^\sigma_\nu\delta^\alpha_\mu\Big)\Big]\right\}\partial_{x\alpha}\partial_{x'\beta}F^{(\mathrm{g})}_{\sigma\rho}(x,x')\bigg|_{x'=x}\,,
\end{aligned}$$

so that

$$
\begin{aligned}
T^{(\mathrm{g})}_{\mu\nu}(x) &= 2\,\frac{\delta\Gamma^{(\mathrm{g})}[D,S]}{\delta g^{\mu\nu}(x)}\bigg|_{g_{\mu\nu}(x)=\eta_{\mu\nu}} \\
&= -\frac{1}{2}\bigg\{-\eta_{\mu\nu}\left(\eta^{\rho\sigma}\eta^{\alpha\beta}-\eta^{\rho\alpha}\eta^{\sigma\beta}+\frac{1}{\xi}\,\eta^{\rho\beta}\eta^{\alpha\sigma}\right) \\
&\qquad + \Big(\delta^\rho_\mu\delta^\sigma_\nu+\delta^\rho_\nu\delta^\sigma_\mu\Big)\eta^{\alpha\beta}+\eta^{\rho\sigma}\Big(\delta^\alpha_\mu\delta^\beta_\nu+\delta^\alpha_\nu\delta^\beta_\mu\Big) \\
&\qquad - \Big[\Big(\delta^\rho_\mu\delta^\alpha_\nu+\delta^\rho_\nu\delta^\alpha_\mu\Big)\eta^{\sigma\beta}+\eta^{\rho\alpha}\Big(\delta^\sigma_\mu\delta^\beta_\nu+\delta^\sigma_\nu\delta^\beta_\mu\Big)\Big] \\
&\qquad + \frac{1}{\xi}\Big[\Big(\delta^\rho_\mu\delta^\beta_\nu+\delta^\rho_\nu\delta^\beta_\mu\Big)\eta^{\sigma\alpha}+\eta^{\rho\beta}\Big(\delta^\sigma_\mu\delta^\alpha_\nu+\delta^\sigma_\nu\delta^\alpha_\mu\Big)\Big]\bigg\}\,\partial_{x\alpha}\partial_{x'\beta}F^{(\mathrm{g})}_{\sigma\rho}(x,x')\bigg|_{x'=x} \\
&= -\frac{1}{2}\bigg\{-\eta_{\mu\nu}\Big[\partial_{x\rho}\partial^\rho_{x'}F^{(\mathrm{g})\sigma}{}_\sigma(x,x')-\partial^\rho_x\partial^\sigma_{x'}F^{(\mathrm{g})}_{\sigma\rho}(x,x')+\frac{1}{\xi}\,\partial^\sigma_x\partial^\rho_{x'}F^{(\mathrm{g})}_{\sigma\rho}(x,x')\Big] \\
&\qquad + \partial_{x\rho}\partial^\rho_{x'}\Big[F^{(\mathrm{g})}_{\mu\nu}(x,x')+F^{(\mathrm{g})}_{\nu\mu}(x,x')\Big]+\Big(\partial_{x\mu}\partial_{x'\nu}+\partial_{x\nu}\partial_{x'\mu}\Big)F^{(\mathrm{g})\rho}{}_\rho(x,x') \\
&\qquad - \Big[\ \partial_{x\nu}\partial^\sigma_{x'}F^{(\mathrm{g})}_{\sigma\mu}(x,x')+\partial_{x\mu}\partial^\sigma_{x'}F^{(\mathrm{g})}_{\sigma\nu}(x,x') \\
&\qquad\quad + \partial^\rho_x\partial_{x'\nu}F^{(\mathrm{g})}_{\mu\rho}(x,x')+\partial^\rho_x\partial_{x'\mu}F^{(\mathrm{g})}_{\nu\rho}(x,x')\Big] \\
&\qquad + \frac{1}{\xi}\Big[\ \partial^\sigma_x\partial_{x'\nu}F^{(\mathrm{g})}_{\sigma\mu}(x,x')+\partial^\sigma_x\partial_{x'\mu}F^{(\mathrm{g})}_{\sigma\nu}(x,x') \\
&\qquad\quad + \partial_{x\nu}\partial^\rho_{x'}F^{(\mathrm{g})}_{\mu\rho}(x,x')+\partial_{x\mu}\partial^\rho_{x'}F^{(\mathrm{g})}_{\nu\rho}(x,x')\Big]\bigg\}\bigg|_{x'=x} \\
&= -\frac{1}{2}\bigg\{-\eta_{\mu\nu}\Big[\partial_{x\rho}\partial^\rho_{x'}F^{(\mathrm{g})\sigma}{}_\sigma(x,x')-\partial^\rho_x\partial^\sigma_{x'}F^{(\mathrm{g})}_{\sigma\rho}(x,x')\Big] \\
&\qquad + \partial_{x\rho}\partial^\rho_{x'}\Big[F^{(\mathrm{g})}_{\mu\nu}(x,x')+F^{(\mathrm{g})}_{\nu\mu}(x,x')\Big]+\Big(\partial_{x\mu}\partial_{x'\nu}+\partial_{x\nu}\partial_{x'\mu}\Big)F^{(\mathrm{g})\rho}{}_\rho(x,x') \\
&\qquad - \Big[\ \partial_{x\mu}\partial^\rho_{x'}F^{(\mathrm{g})}_{\rho\nu}(x,x')+\partial_{x\nu}\partial^\rho_{x'}F^{(\mathrm{g})}_{\rho\mu}(x,x') \\
&\qquad\quad + \partial^\rho_x\partial_{x'\mu}F^{(\mathrm{g})}_{\nu\rho}(x,x')+\partial^\rho_x\partial_{x'\nu}F^{(\mathrm{g})}_{\mu\rho}(x,x')\Big] \\
&\qquad - \Big[\partial_{x\mu}F^{(AB)}_\nu(x,x')+\partial_{x\nu}F^{(AB)}_\mu(x,x')+\partial_{x'\mu}F^{(BA)}_\nu(x,x')+\partial_{x'\nu}F^{(BA)}_\mu(x,x')\Big]\bigg\}\bigg|_{x'=x} \\
&= -\frac{1}{2}\bigg\{-\eta_{\mu\nu}\Big[\partial_{x\rho}\partial^\rho_{x'}F^{(\mathrm{g})\sigma}{}_\sigma(x,x')-\partial^\rho_x\partial^\sigma_{x'}F^{(\mathrm{g})}_{\sigma\rho}(x,x')\Big]
\end{aligned}
$$

$$
\begin{aligned}
&+ \partial_{x\rho}\partial^{\rho}_{x'}\Big[F^{(\mathrm{g})}_{\mu\nu}(x,x') + F^{(\mathrm{g})}_{\nu\mu}(x,x')\Big] + \Big(\partial_{x\mu}\partial_{x'\nu} + \partial_{x\nu}\partial_{x'\mu}\Big)F^{(\mathrm{g})\rho}{}_{\rho}(x,x') \\
&- \Big[\ \partial_{x\mu}\partial^{\rho}_{x'}F^{(\mathrm{g})}_{\rho\nu}(x,x') + \partial_{x\nu}\partial^{\rho}_{x'}F^{(\mathrm{g})}_{\rho\mu}(x,x') \\
&\quad + \partial^{\rho}_{x}\partial_{x'\mu}F^{(\mathrm{g})}_{\nu\rho}(x,x') + \partial^{\rho}_{x}\partial_{x'\nu}F^{(\mathrm{g})}_{\mu\rho}(x,x')\Big]\Big\}\Big|_{x'=x} \\
&+ \Big[\partial_{x\mu}F^{(AB)}_{\nu}(x,x') + \partial_{x\nu}F^{(AB)}_{\mu}(x,x')\Big]\Big|_{x'=x}
\end{aligned}
\tag{B.27}
$$

The last line consists only of correlators involving an auxiliary field. In the full theory, where they are free, we can insert their corresponding solutions and obtain:

$$
\begin{aligned}
&\partial_{x\mu}F^{(AB)}_{\nu}(x,x')\Big|_{x'=x} \\
&= \int_{\boldsymbol{p}} \left(\delta^0_\mu \frac{\partial}{\partial x^0} - \mathrm{i}\,\delta^i_\mu p_i\right)\left[\frac{1}{2} + n^{(\mathrm{g})}(\boldsymbol{p})\right] \\
&\qquad \cdot \left[\delta^0_\nu \sin(|\boldsymbol{p}|(x^0 - x'^0)) + \mathrm{i}\,\delta^j_\nu \frac{p_j}{|\boldsymbol{p}|}\cos(|\boldsymbol{p}|(x^0 - x'^0))\right]\Bigg|_{x'^0=x^0} \\
&= \int_{\boldsymbol{p}} |\boldsymbol{p}|\left[\frac{1}{2} + n^{(\mathrm{g})}(\boldsymbol{p})\right] \\
&\qquad \cdot \Bigg\{\delta^0_\mu\left[\delta^0_\nu \cos(|\boldsymbol{p}|(x^0 - x'^0)) - \mathrm{i}\,\delta^i_\nu \frac{p_i}{|\boldsymbol{p}|}\sin(|\boldsymbol{p}|(x^0 - x'^0))\right] \\
&\qquad\quad - \mathrm{i}\,\delta^i_\mu \frac{p_i}{|\boldsymbol{p}|}\left[\delta^0_\nu \cos(|\boldsymbol{p}|(x^0 - x'^0)) + \mathrm{i}\,\delta^j_\nu \frac{p_j}{|\boldsymbol{p}|}\cos(|\boldsymbol{p}|(x^0 - x'^0))\right]\Bigg\}\Bigg|_{x'^0=x^0} \\
&= \int_{\boldsymbol{p}} |\boldsymbol{p}|\left[\frac{1}{2} + n^{(\mathrm{g})}(\boldsymbol{p})\right]\left(\delta^0_\mu\delta^0_\nu + \delta^i_\mu\delta^j_\nu \frac{p_i p_j}{\boldsymbol{p}^2}\right)
\end{aligned}
\tag{B.28}
$$

and

$$
\begin{aligned}
&\partial_{x'\nu}F^{(BA)}_{\mu}(x,x')\Big|_{x'=x} \\
&= -\int_{\boldsymbol{p}} \left(\delta^0_\nu \frac{\partial}{\partial x'^0} + \mathrm{i}\,\delta^i_\nu p_i\right)\left[\frac{1}{2} + n^{(\mathrm{g})}(\boldsymbol{p})\right] \\
&\qquad \cdot \left[\delta^0_\mu \sin(|\boldsymbol{p}|(x^0 - x'^0)) + \mathrm{i}\,\delta^j_\mu \frac{p_j}{|\boldsymbol{p}|}\cos(|\boldsymbol{p}|(x^0 - x'^0))\right]\Bigg|_{x'^0=x^0} \\
&= -\int_{\boldsymbol{p}} |\boldsymbol{p}|\left[\frac{1}{2} + n^{(\mathrm{g})}(\boldsymbol{p})\right] \\
&\qquad \cdot \Bigg\{\delta^0_\nu\left[-\delta^0_\mu \cos(|\boldsymbol{p}|(x^0 - x'^0)) + \mathrm{i}\,\delta^i_\mu \frac{p_i}{|\boldsymbol{p}|}\sin(|\boldsymbol{p}|(x^0 - x'^0))\right] \\
&\qquad\quad + \mathrm{i}\,\delta^i_\nu \frac{p_i}{|\boldsymbol{p}|}\left[\delta^0_\mu \cos(|\boldsymbol{p}|(x^0 - x'^0)) + \mathrm{i}\,\delta^j_\mu \frac{p_j}{|\boldsymbol{p}|}\cos(|\boldsymbol{p}|(x^0 - x'^0))\right]\Bigg\}\Bigg|_{x'^0=x^0}
\end{aligned}
$$

$$= \int_{\boldsymbol{p}} |\boldsymbol{p}| \left[\frac{1}{2} + n^{(\mathrm{g})}(\boldsymbol{p})\right] \left(\delta^0_\mu \delta^0_\nu + \delta^i_\mu \delta^j_\nu \frac{p_i p_j}{\boldsymbol{p}^2}\right) \tag{B.29}$$

so that

$$\left[\partial_{x\mu} F^{(AB)}_{\nu}(x,x') + \partial_{x\nu} F^{(AB)}_{\mu}(x,x')\right]\Big|_{x'=x} = 2\int_{\boldsymbol{p}} |\boldsymbol{p}| \left[\frac{1}{2} + n^{(\mathrm{g})}(\boldsymbol{p})\right] \left(\delta^0_\mu \delta^0_\nu + \delta^i_\mu \delta^j_\nu \frac{p_i p_j}{\boldsymbol{p}^2}\right) . \tag{B.30}$$

Note that this contribution of the gauge fixing part to the energy-momentum tensor exactly cancels the ghost contribution (B.22), so that the physical, gauge invariant photon energy density is given by:

$$\begin{aligned}
T^{(\mathrm{g})\mathrm{phys}}_{\mu\nu}(x) &= T^{(\mathrm{g})}_{\mu\nu}(x) + T^{(\mathrm{gh})}_{\mu\nu}(x) \\
&= \frac{1}{2}\Bigg\{\eta_{\mu\nu}\left[\partial_{x\rho}\partial^\rho_{x'} F^{(\mathrm{g})\sigma}{}_\sigma(x,x') - \partial^\rho_x \partial^\sigma_{x'} F^{(\mathrm{g})}_{\sigma\rho}(x,x')\right] \\
&\qquad - \partial_{x\rho}\partial^\rho_{x'}\left[F^{(\mathrm{g})}_{\mu\nu}(x,x') + F^{(\mathrm{g})}_{\nu\mu}(x,x')\right] - \left(\partial_{x\mu}\partial_{x'\nu} + \partial_{x\nu}\partial_{x'\mu}\right) F^{(\mathrm{g})\rho}{}_\rho(x,x') \\
&\qquad + \partial_{x\mu}\partial^\rho_{x'} F^{(\mathrm{g})}_{\rho\nu}(x,x') + \partial_{x\nu}\partial^\rho_{x'} F^{(\mathrm{g})}_{\rho\mu}(x,x') \\
&\qquad + \partial^\rho_x \partial_{x'\mu} F^{(\mathrm{g})}_{\nu\rho}(x,x') + \partial^\rho_x \partial_{x'\nu} F^{(\mathrm{g})}_{\mu\rho}(x,x')\Bigg]\Bigg\}\Bigg|_{x'=x} .
\end{aligned} \tag{B.31}$$

Note that the physical energy-momentum tensor of the photon sector is traceless, i. e. $T^{(\mathrm{g})\mathrm{phys}\mu}{}_\mu(x) = 0$. This is because the photon sector is conformal since there is no mass scale.

### Belinfante–Rosenfeld Energy-Momentum Tensor

The other way to obtain the energy-momentum tensor, as already mentioned earlier, is via Noether's theorem as the conserved current corresponding to the symmetry of the system under spacetime translations (which yields the canonical energy momentum tensor). One advantage of this method is that, since one starts with the original action in terms of quantum field operators, it is not necessary to fix a gauge in the first place, and hence no Faddeev-Popov ghosts appear.[15] In the end, however, we are interested in obtaining an expression in terms of correlation functions instead of quantum field operators, so that one has to calculate expectation values. In a truncated theory, however, the connection between the correlation functions obtained by forming expectation values of quantum field operators and those the 2PI effective action depends on is not at all clear. While for the exact theory, the expressions obtained via both methods have to agree, this is not necessarily the case for any truncated theory. We will nevertheless show that the expressions do agree for the exact theory, where the expectation value of two quantum field operators does correspond

[15]It is clear, after all, that it should be possible to derive the energy-momentum tensor without having to fix a gauge since it is a physical quantity.

to the physical propagator which solves the corresponding EoM in the 2PI formalism. This is a good check for our previous calculation involving the 2PI effective action.

We have:[16]

$$\widehat{T}^{(\mathrm{g})}_{\mathrm{c}\,\mu\nu}(x) = \frac{\partial\mathcal{L}}{\partial\Big(\partial^\nu \widehat{A}_\rho(x)\Big)}\,\partial_\mu \widehat{A}_\rho(x) \tag{B.32}$$

where the photon fields are now operators. This energy-momentum tensor, however, is neither symmetric nor gauge invariant. In order to obtain a symmetric, gauge invariant energy-momentum tensor which equals the one obtained from the Hilbert energy-momentum tensor, we have to construct the Belinfante–Rosenfeld energy-momentum tensor. It reads:

$$\widehat{T}^{(\mathrm{g})}_{\mathrm{BR}\,\mu\nu}(x) = \widehat{F}_\mu{}^\rho(x)\widehat{F}_{\rho\nu}(x) + \frac{1}{4}\,\eta_{\mu\nu}\widehat{F}_{\rho\sigma}(x)\widehat{F}^{\rho\sigma}(x)\,. \tag{B.33}$$

That it is symmetric is clear by inspection, and it is manifestly gauge invariance since it is constructed solely from the gauge invariant field-strength tensor. Its expectation value is then given by:

$$\begin{aligned}
T^{(\mathrm{g})}_{\mathrm{BR}\,\mu\nu}(x) &= \Big\langle \widehat{T}^{(\mathrm{g})}_{\mathrm{BR}\,\mu\nu}(x)\Big\rangle \\
&= \Big\langle \widehat{F}_\mu{}^\rho(x)\widehat{F}_{\rho\nu}(x) + \frac{1}{4}\,\eta_{\mu\nu}\widehat{F}_{\rho\sigma}(x)\widehat{F}^{\rho\sigma}(x)\Big\rangle \\
&= \Big\langle \Big[\partial_{x\mu}\widehat{A}^\rho(x) - \partial^\rho_x \widehat{A}_\mu(x)\Big]\Big[\partial_{x\rho}\widehat{A}_\nu(x) - \partial_{x\nu}\widehat{A}_\rho(x)\Big] \\
&\qquad + \frac{1}{4}\,\eta_{\mu\nu}\Big[\partial_{x\rho}\widehat{A}_\sigma(x) - \partial_{x\sigma}\widehat{A}_\rho(x)\Big]\Big[\partial^\rho_x \widehat{A}^\sigma(x) - \partial^\sigma_x \widehat{A}^\rho(x)\Big]\Big\rangle \\
&= \partial_{x\mu}\partial_{x\rho}F^{(\mathrm{g})\rho}{}_\nu(x,x) - \partial_{x\mu}\partial_{x\nu}F^{(\mathrm{g})\rho}{}_\rho(x,x) - \partial^\rho_x\partial_{x\rho}F^{(\mathrm{g})}_{\mu\nu}(x,x') + \partial^\rho_x\partial_{x\nu}F^{(\mathrm{g})}_{\mu\rho}(x,x) \\
&\quad + \frac{1}{2}\,\eta_{\mu\nu}\Big[\partial_{x\rho}\partial^\rho_x F^{(\mathrm{g})}{}_\sigma{}^\sigma(x,x) - \partial_{x\rho}\partial^\sigma_x F^{(\mathrm{g})}{}_\sigma{}^\rho(x,x)\Big] \\
&= \partial_{x\mu}\partial^\rho_x F^{(\mathrm{g})}_{\rho\nu}(x,x) - \partial_{x\mu}\partial_{x\nu}F^{(\mathrm{g})\rho}{}_\rho(x,x) - \partial^\rho_x\partial_{x\rho}F^{(\mathrm{g})}_{\mu\nu}(x,x) + \partial^\rho_x\partial_{x\nu}F^{(\mathrm{g})}_{\mu\rho}(x,x) \\
&\quad + \frac{1}{2}\,\eta_{\mu\nu}\Big[\partial_{x\rho}\partial^\rho_x F^{(\mathrm{g})}{}_\sigma{}^\sigma(x,x) - \partial^\rho_x\partial^\sigma_x F^{(\mathrm{g})}_{\sigma\rho}(x,x)\Big]\,.
\end{aligned} \tag{B.34}$$

Since the statistical function is evaluated at equal spacetime points, this expression is ill-defined as it stands and therefore has to be regularized. We hence apply the point-splitting procedure (B.24) and obtain:

$$\begin{aligned}
&T^{(\mathrm{g})}_{\mathrm{BR}\,\mu\nu}(x) \\
&\quad = \frac{1}{2}\Big\langle \hat{F}_\mu{}^\rho(x)\hat{F}_{\rho\nu}(x') + \frac{1}{4}\,\eta_{\mu\nu}\hat{F}_{\rho\sigma}(x)\hat{F}^{\rho\sigma}(x') + \hat{F}_\mu{}^\rho(x')\hat{F}_{\rho\nu}(x) + \frac{1}{4}\,\eta_{\mu\nu}\hat{F}_{\rho\sigma}(x')\hat{F}^{\rho\sigma}(x)\Big\rangle\Big|_{x'=x} \\
&\quad = \frac{1}{2}\Big\{\partial_{x\mu}\partial^\rho_{x'}F^{(\mathrm{g})}_{\rho\nu}(x,x') + \partial_{x'\mu}\partial^\rho_x F^{(\mathrm{g})}_{\nu\rho}(x,x') - \partial_{x\mu}\partial_{x'\nu}F^{(\mathrm{g})\rho}{}_\rho(x,x') - \partial_{x'\mu}\partial_{x\nu}F^{(\mathrm{g})\rho}{}_\rho(x,x')
\end{aligned}$$

[16] Only in this section we use hats to indicate operators.

$$
\begin{aligned}
&- \partial_{x\rho}\partial^{\rho}_{x'} F^{(\mathrm{g})}_{\mu\nu}(x,x') - \partial_{x'\rho}\partial^{\rho}_{x} F^{(\mathrm{g})}_{\nu\mu}(x,x') + \partial^{\rho}_{x}\partial_{x'\nu} F^{(\mathrm{g})}_{\mu\rho}(x,x') + \partial^{\rho}_{x'}\partial_{x\nu} F^{(\mathrm{g})}_{\rho\mu}(x,x') \\
&+ \frac{1}{2}\,\eta_{\mu\nu}\Big[\partial_{x\rho}\partial^{\rho}_{x'} F^{(\mathrm{g})\sigma}{}_{\sigma}(x,x') + \partial_{x'\rho}\partial^{\rho}_{x} F^{(\mathrm{g})\sigma}{}_{\sigma}(x,x') - \partial^{\rho}_{x}\partial^{\sigma}_{x'} F^{(\mathrm{g})}_{\sigma\rho}(x,x') - \partial^{\rho}_{x'}\partial^{\sigma}_{x} F^{(\mathrm{g})}_{\rho\sigma}(x,x')\Big]\Big\}\Big|_{x'=x} \\
= {} & \frac{1}{2}\Big\{\eta_{\mu\nu}\Big[\partial_{x\rho}\partial^{\rho}_{x'} F^{(\mathrm{g})\sigma}{}_{\sigma}(x,x') - \partial^{\rho}_{x}\partial^{\sigma}_{x'} F^{(\mathrm{g})}_{\sigma\rho}(x,x')\Big] \\
&- \partial_{x\rho}\partial^{\rho}_{x'}\Big[F^{(\mathrm{g})}_{\mu\nu}(x,x') + F^{(\mathrm{g})}_{\nu\mu}(x,x')\Big] - \Big(\partial_{x\mu}\partial_{x'\nu} + \partial_{x\nu}\partial_{x'\mu}\Big) F^{(\mathrm{g})\rho}{}_{\rho}(x,x') \\
&+ \partial_{x\mu}\partial^{\rho}_{x'} F^{(\mathrm{g})}_{\rho\nu}(x,x') + \partial_{x\nu}\partial^{\rho}_{x'} F^{(\mathrm{g})}_{\rho\mu}(x,x') + \partial^{\rho}_{x}\partial_{x'\mu} F^{(\mathrm{g})}_{\nu\rho}(x,x') + \partial^{\rho}_{x}\partial_{x'\nu} F^{(\mathrm{g})}_{\mu\rho}(x,x')\Big\}\Big|_{x'=x} \\
= {} & T^{(\mathrm{g})\mathrm{phys}}_{\mu\nu}(x)\,,
\end{aligned}
\tag{B.35}
$$

so the Belinfante–Rosenfeld energy-momentum tensor constructed from the canonical energy-momentum tensor with the given point-splitting procedure yields the same result as Eq. (B.31) obtained by varying the 2PI effective action.

**Energy Density** The energy density is then given by:

$$
\begin{aligned}
T^{(\mathrm{g})\mathrm{phys}}_{00}(x) = \frac{1}{2}\Big[&-\partial_{xi}\partial^{i}_{x'} F^{(\mathrm{g})}_{00}(x,x') - (\partial^{0}_{x}\partial^{0}_{x'} - \partial_{xj}\partial^{j}_{x'}) F^{(\mathrm{g})i}{}_{i}(x,x') - \partial^{i}_{x}\partial^{j}_{x'} F^{(\mathrm{g})}_{ji}(x,x') \\
&+ \partial^{0}_{x}\partial^{i}_{x'} F^{(\mathrm{g})}_{i0}(x,x') + \partial^{i}_{x}\partial^{0}_{x'} F^{(\mathrm{g})}_{0i}(x,x')\Big]\Big|_{x'=x}\,,
\end{aligned}
$$

or in terms of the isotropic components:[17]

$$
\begin{aligned}
&\varepsilon^{(\mathrm{g})\mathrm{phys}}(t) \\
&\quad = -\frac{1}{2}\int_p\Big\{\Big(\frac{\partial}{\partial t}\frac{\partial}{\partial t'} + p^2\Big)\Big[2F^{(\mathrm{g})}_{\mathrm{T}}(t,t';p) + F^{(\mathrm{g})}_{\mathrm{L}}(t,t';p)\Big] - p^2\Big[F^{(\mathrm{g})}_{\mathrm{S}}(t,t';p) + F^{(\mathrm{g})}_{\mathrm{L}}(t,t';p)\Big] \\
&\qquad - p\Big[\frac{\partial}{\partial t}\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';p) - \frac{\partial}{\partial t'}\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';p)\Big]\Big\}\Big|_{t'=t}\,.
\end{aligned}
\tag{B.36}
$$

[17]If the Ward identity could be applied, we could simplify the expression by subtracting $\partial^{\mu}_{x}\partial^{\nu}_{x'} F^{(\mathrm{g})}_{\mu\nu}(x,x') = 0$, or in terms of isotropic components:

$$
\frac{\partial}{\partial t}\frac{\partial}{\partial t'} F^{(\mathrm{g})}_{\mathrm{S}}(t,t';p) + p\Big[\frac{\partial}{\partial t}\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';p) - \frac{\partial}{\partial t'}\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';p)\Big] - p^2 F^{(\mathrm{g})}_{\mathrm{L}}(t,t';p) = 0\,,
$$

from the energy density to obtain:

$$
\begin{aligned}
\varepsilon^{(\mathrm{g})\mathrm{phys}}(t) = \frac{1}{2}\int_p\Big\{&\Big(\frac{\partial}{\partial t}\frac{\partial}{\partial t'} + p^2\Big)\Big[F^{(\mathrm{g})}_{\mathrm{S}}(t,t';p) - 2F^{(\mathrm{g})}_{\mathrm{T}}(t,t';p) - F^{(\mathrm{g})}_{\mathrm{L}}(t,t';p)\Big] \\
&+ p\Big(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\Big)\Big[\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(t,t';p) + \widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(t,t';p)\Big]\Big\}\Big|_{t'=t}\,.
\end{aligned}
$$

For a free photon gas, we obtain:

$$\begin{aligned}\varepsilon_0^{(\mathrm{g})\mathrm{phys}} &= \int_p \left(\frac{\partial}{\partial t}\frac{\partial}{\partial t'} + p^2\right) F_{\mathrm{T}}^{(\mathrm{g})}(t,t';p)\Big|_{t'=t} = 2\int_p p\left[\frac{1}{2} + n^{(\mathrm{g})}(p)\right] \\ &= \frac{1}{\pi^2}\int_0^\infty \mathrm{d}p\, p^3 \left[\frac{1}{2} + n^{(\mathrm{g})}(p)\right] .\end{aligned} \tag{B.37}$$

The energy density of a free photon gas is of course time-independent and depends on the photon distribution $n^{(\mathrm{g})}$. For some common distribution functions we have:

- Vacuum: $n^{(\mathrm{g})}(p) = 0$:

$$\varepsilon_{0,\mathrm{vac}}^{(\mathrm{g})} = \lim_{\Lambda\to\infty} \frac{\Lambda^4}{8\pi^2} \to \infty.$$

  The vacuum energy density diverges. If, however, a finite momentum cutoff $\Lambda$ is introduced, the result becomes finite and depends on a single parameter, the cutoff. This then resembles a cosmological constant.
- Thermal equilibrium: $n^{(\mathrm{g})}(p) = n_{\mathrm{BE}}(p) = 1\Big/\left(\mathrm{e}^{p/T} - 1\right)$:

$$\varepsilon_{0,\mathrm{therm}}^{(\mathrm{g})} = \varepsilon_{0,\mathrm{vac}}^{(\mathrm{g})} + \frac{\pi^2 T^4}{15} .$$

  Thermal equilibrium depends on a single parameter, the temperature $T$, and is the energy density of a free boson gas with a degeneracy factor of two, corresponding to the two spin states of the photon.
- Gaussian or "tsunami": $n^{(\mathrm{g})}(p) = n_{\mathrm{Gauss}}(p) = A\exp(-(p-\overline{p})^2/(2\sigma^2))$:

$$\begin{aligned}&\varepsilon_{0,\mathrm{Gauss}}^{(\mathrm{g})} \\ &\quad = \varepsilon_{0,\mathrm{vac}}^{(\mathrm{g})} + \frac{A\,\sigma}{2\pi^2}\left\{\sqrt{2\pi}\,\overline{p}\,(\overline{p}^2 + 3\sigma^2)\left[1 + \mathrm{erf}\left(\frac{\overline{p}}{\sqrt{2}\,\sigma}\right)\right] + 2\sigma\,(\overline{p}^2 + 2\sigma^2)\,\mathrm{e}^{-\overline{p}^2/(2\sigma^2)}\right\} .\end{aligned}$$

  It depends on three parameters, the amplitude $A$, the mean $\overline{p}$, and the width $\sigma$.
- Homogeneous "stream": $n^{(\mathrm{g})}(p) = n_{\mathrm{stream}}(p) = p_0\,\delta(p-\overline{p})$:

$$\varepsilon_{0,\mathrm{stream}}^{(\mathrm{g})} = \varepsilon_{0,\mathrm{vac}}^{(\mathrm{g})} + \frac{p_0\,\overline{p}^3}{\pi^2} .$$

  It depends on two parameters, the momentum $p_0$ and the stream momentum $\overline{p}$.

**Pressure Density** Since the photon part of the energy-momentum tensor is conformal, the pressure density is just one third of the energy density, i. e.

$$p^{(\mathrm{g})\mathrm{phys}}(t) = \frac{1}{3}\,\varepsilon^{(\mathrm{g})\mathrm{phys}}(t) . \tag{B.38}$$

### B.4.6 Fermion Part

We have:[18]

$$
\begin{aligned}
-\,\mathrm{i}\,\mathrm{Tr}(S_0^{-1}S) &= -\,\mathrm{i}\int_{y,z}\sqrt{-g(y)}\,\sqrt{-g(z)}\;\mathrm{tr}\Big(S_0^{-1}(y,z)S(z,y)\Big)\\
&= -\int_{y,z}\sqrt{-g(y)}\,\sqrt{-g(z)}\;\mathrm{tr}\bigg(\Big\{\Big[\mathrm{i}\,\gamma^\rho(y)\,\partial_{y\sigma}-m^{(\mathrm{f})}\Big]\,\delta^4(y-z)\Big\}\,S(z,y)\bigg)\\
&= -\int_{y,z}\sqrt{-g(y)}\,\sqrt{-g(z)}\,\delta^4(y-z)\;\mathrm{tr}\Big(\Big[-\,\mathrm{i}\,\gamma^\rho(y)\,\partial_{y\rho}-m^{(\mathrm{f})}\Big]\,S(z,y)\Big)\\
&= \int_{z}\sqrt{-g(z)}\;\mathrm{tr}\Big(\Big[\mathrm{i}\,\gamma^\rho(y)\,\partial_{z'\rho}+m^{(\mathrm{f})}\Big]S(z,z')\Big)\Big|_{z'=z}\,.
\end{aligned}
$$

Note that, as for the photons, the first term of the fermionic part of the effective action does not contribute to the energy density since it is independent of the metric. Further note that, since the derivative in the free inverse fermion propagator acts on a delta distribution, we integrated by parts so that the derivative acts on the full fermion propagator and the delta distribution can be used to solve one of the integrals.

Note that the gamma matrices depend on spacetime when the metric does, which follows from the definition of their Clifford algebra, given by

$$
\Big\{\gamma^\mu(x),\gamma^\nu(x)\Big\} = 2g^{\mu\nu}(x)\,. \tag{B.39}
$$

It follows that the gamma matrices depend in a nontrivial way on the metric.[19] Its dependence can be found by making the following ansatz for the variation of the metric [Sor77]:

$$
\delta\gamma^\mu(x) = A^\mu{}_\nu(x)\gamma^\nu(x)\,,
$$

[18] Note that to be precise, one has to replace the derivative $\partial_\mu$ by the covariant derivative $\nabla_\mu$. The covariant derivative depends on a connection, whose variation with respect to the metric does not vanish in general. However, it turns out that the result of varying the connection with respect to the metric results in derivatives of gamma matrices [Sor77], which vanish in Minkowski spacetime. One can therefore ignore the contribution of the connection and just work with the usual derivative in the first place.

[19] Gamma matrices can only be defined in a flat spacetime. In order to formulate fermionic theories in a general spacetime, it is necessary to introduce *tetrad* (or *vierbein*) fields $\mathsf{e}_a^\mu(x)$. It is them which depend on spacetime, and by contracting them with the gamma matrices, we obtain "effectively" spacetime dependent gamma matrices $\gamma^\mu(x)=\gamma^a\mathsf{e}_a^\mu(x)$. It then follows that

$$
\big\{\gamma^\mu(x),\gamma^\nu(y)\big\} = \big\{\mathsf{e}_a^\mu(x)\gamma^a,\mathsf{e}_b^\nu(x)\gamma^b\big\} = \mathsf{e}_a^\mu(x)\mathsf{e}_b^\nu(x)\{\gamma^a,\gamma^b\} = 2\mathsf{e}_a^\mu(x)\mathsf{e}_b^\nu(x)\eta^{ab} = 2g^{\mu\nu}(x)\,.
$$

The tetrads also depend nontrivially on the metric since $g_{\mu\nu}=\mathsf{e}_\mu^a\mathsf{e}_\nu^b\eta_{ab}$. From $\delta g_{\mu\nu}=\big[(\delta\mathsf{e}_\mu^a)\mathsf{e}_\nu^b+\mathsf{e}_\mu^a\delta\mathsf{e}_\nu^b\big]\eta_{ab}=(\delta_\mu^\rho\delta_c^a\mathsf{e}_\nu^b+\delta_\nu^\rho\delta_c^b\mathsf{e}_\mu^a)\delta\mathsf{e}_\rho^c\eta_{ab}$, so that

$$
\frac{\delta g_{\mu\nu}}{\delta\mathsf{e}_\rho^c} = (\delta_\mu^\rho\delta_c^a\mathsf{e}_\nu^b+\delta_\nu^\rho\delta_c^b\mathsf{e}_\mu^a)\eta_{ab} = \delta_\mu^\rho\mathsf{e}_\nu^b\eta_{cb}+\delta_\nu^\rho\mathsf{e}_\mu^a\eta_{ac} = \delta_\mu^\rho\mathsf{e}_{\nu c}+\delta_\nu^\rho\eta_{\mu c}\,.
$$

i. e. by assuming that the variation of the gamma matrices is linear. Then:

$$\begin{aligned}
2\delta g^{\mu\nu}(x) &= \delta\Big\{\gamma^\mu(x),\gamma^\nu(x)\Big\} \\
&= \Big\{\delta\gamma^\mu(x),\gamma^\nu(x)\Big\} + \Big\{\gamma^\mu(x),\delta\gamma^\nu(x)\Big\} \\
&= A^\mu{}_\rho(x)\Big\{\gamma^\rho(x),\gamma^\nu(x)\Big\} + A^\nu{}_\rho(x)\Big\{\gamma^\mu(x),\gamma^\rho(x)\Big\} \\
&= 2A^\mu{}_\rho(x)g^{\rho\nu}(x) + 2A^\nu{}_\rho(x)g^{\mu\rho}(x) \\
&= 2\Big[A^{\mu\nu}(x) + A^{\nu\mu}(x)\Big]\,,
\end{aligned}$$

so that

$$\delta g^{\mu\nu}(x) = A^{\mu\nu}(x) + A^{\nu\mu}(x)\,.$$

Further assuming that $A^{\mu\nu}$ is symmetric, it follows that[20]

$$\delta\gamma^\mu(x) = \frac{1}{2}\,\delta g^\mu{}_\nu(x)\gamma^\nu(x)\,.$$

It can then easily be seen that with

$$\frac{\delta\gamma^\rho(y)}{\delta g^{\mu\nu}(x)} = \frac{1}{4}\left[\delta^\rho_\mu\gamma_\nu(y) + \delta^\rho_\nu\gamma_\mu(y)\right]\delta^4(y-x)\,,$$

one obtains

$$\begin{aligned}
&\frac{\delta}{\delta g^{\mu\nu}(x)}\big\{\gamma^\rho(y),\gamma^\sigma(y)\big\} \\
&\quad= \frac{\delta\gamma^\rho(y)}{\delta g^{\mu\nu}(x)}\,\gamma^\sigma(y) + \gamma^\rho(y)\,\frac{\delta\gamma^\sigma(y)}{\delta g^{\mu\nu}(x)} + \frac{\delta\gamma^\sigma(y)}{\delta g^{\mu\nu}(x)}\,\gamma^\rho(y) + \gamma^\sigma(y)\,\frac{\delta\gamma^\rho(y)}{\delta g^{\mu\nu}(x)} \\
&\quad= \frac{1}{4}\bigg\{\ \Big[\delta^\rho_\mu\gamma_\nu(y) + \delta^\rho_\nu\gamma_\mu(y)\Big]\gamma^\sigma(y) + \gamma^\rho(y)\Big[\delta^\sigma_\mu\gamma_\nu(y) + \delta^\sigma_\nu\gamma_\mu(y)\Big] \\
&\qquad\quad + \Big[\delta^\sigma_\mu\gamma_\nu(y) + \delta^\sigma_\nu\gamma_\mu(y)\Big]\gamma^\rho(y) + \gamma^\sigma(y)\Big[\delta^\rho_\mu\gamma_\nu(y) + \delta^\rho_\nu\gamma_\mu(y)\Big]\bigg\}\,\delta^4(y-x) \\
&\quad= \frac{1}{4}\bigg[\ \delta^\rho_\mu\Big\{\gamma_\nu(y),\gamma^\sigma(y)\Big\} + \delta^\rho_\nu\Big\{\gamma_\mu(y),\gamma^\sigma(y)\Big\} \\
&\qquad\quad + \delta^\sigma_\mu\Big\{\gamma^\rho(y),\gamma_\nu(y)\Big\} + \delta^\rho_\nu\Big\{\gamma^\rho(y),\gamma_\mu(y)\Big\}\bigg]\,\delta^4(y-x) \\
&\quad= \frac{1}{2}\left(\delta^\rho_\mu\delta^\sigma_\nu + \delta^\rho_\nu\delta^\sigma_\mu + \delta^\sigma_\mu\delta^\rho_\nu + \delta^\sigma_\nu\delta^\rho_\mu\right)\delta^4(y-x) \\
&\quad= \left(\delta^\rho_\mu\delta^\sigma_\nu + \delta^\sigma_\mu\delta^\rho_\nu\right)\delta^4(y-x) \\
&\quad= 2\,\frac{\delta g^{\rho\sigma}(y)}{\delta g^{\mu\nu}(x)}\,.
\end{aligned}$$

[20] Another way to derive the variation of the gamma matrices is to work in the tetrad formalism. We then have $g_{\mu\nu} = \mathsf{e}^a_\mu\mathsf{e}^b_\nu\eta_{ab}$, i. e. $\big\{\gamma^\mu,\gamma^\nu\big\} = 2\mathsf{e}^a_\mu\mathsf{e}^b_\nu\eta_{ab}$ and thus, by doing the same steps as in the main text, $\big[(\delta\mathsf{e}^\mu_a)\mathsf{e}^\nu_b + \mathsf{e}^\mu_a\delta\mathsf{e}^\nu_b\big]\eta^{ab} = A^{\mu\nu} + A^{\nu\mu}$, so that we can set $(\delta\mathsf{e}^\mu_a)\mathsf{e}^\nu_b\eta^{ab} = A^{\mu\nu}$. It then follows that $\delta\gamma^\mu = (\delta\mathsf{e}^\mu_a)\mathsf{e}_{b\nu}\eta^{ab}\gamma^\nu$ and $\delta g_{\mu\nu} = \big[(\delta\mathsf{e}^a_\mu)\mathsf{e}^b_\nu + \mathsf{e}^a_\mu\delta\mathsf{e}^b_\nu\big]\eta_{ab} = A_{\mu\nu} + A_{\nu\mu}$.

It then follows that

$$
\begin{aligned}
&\frac{\delta\Gamma^{(\mathrm{f})}[S,g]}{\delta g^{\mu\nu}(x)} \\
&\quad = \frac{1}{2}\int_z \sqrt{-g(z)}\,\delta^4(z-x)\left\{-g_{\mu\nu}(z)\,\mathrm{tr}\Big(\Big[\mathrm{i}\,\gamma^\rho(z')\partial_{z'\rho}+m^{(\mathrm{f})}\Big]S(z,z')\Big)\right. \\
&\qquad\qquad \left.+\frac{\mathrm{i}}{2}\,\mathrm{tr}\Big(\Big[\gamma_\nu(z')\partial_{z'\mu}+\gamma_\mu(z')\partial_{z'\nu}\Big]S(z,z')\Big)\right\}\Bigg|_{z'=z} \\
&\quad = \frac{1}{2}\left\{-g_{\mu\nu}(x)\,\mathrm{tr}\Big(\Big[\mathrm{i}\,\gamma^\rho(z')\partial_{x'\rho}+m^{(\mathrm{f})}\Big]S(x,x')\Big)+\frac{\mathrm{i}}{2}\,\mathrm{tr}\Big(\Big[\gamma_\nu(x)\partial_{x'\mu}+\gamma_\mu\partial_{x'\nu}\Big]S(x,x')\Big)\right\}\Bigg|_{x'=x} \\
&\quad = 2\left\{-g_{\mu\nu}(x)\Big[\mathrm{i}\,\partial_{x'\rho}F_{\mathrm{V}}^{(\mathrm{f})\rho}(x,x')+m^{(\mathrm{f})}F_{\mathrm{S}}^{(\mathrm{f})}(x,x')\Big]+\frac{\mathrm{i}}{2}\Big[\partial_{x'\mu}F_{\mathrm{V}}^{(\mathrm{f})\nu}(x,x')+\partial_{x'\nu}F_{\mathrm{V}}^{(\mathrm{f})\mu}(x,x')\Big]\right\}\Bigg|_{x'=x},
\end{aligned}
$$

so that

$$
\begin{aligned}
&T_{\mu\nu}^{(\mathrm{f})}(x) \\
&\quad = 2\,\frac{\delta\Gamma^{(\mathrm{f})}[S,g]}{\delta g^{\mu\nu}(x)}\Bigg|_{g_{\mu\nu}(x)=\eta_{\mu\nu}} \\
&\quad = 4\left\{-\eta_{\mu\nu}\Big[\mathrm{i}\,\partial_{x'\rho}F_{\mathrm{V}}^{(\mathrm{f})\rho}(x,x')+m^{(\mathrm{f})}F_{\mathrm{S}}^{(\mathrm{f})}(x,x')\Big]+\frac{\mathrm{i}}{2}\Big[\partial_{x'\mu}F_{\mathrm{V}}^{(\mathrm{f})}{}_{\nu}(x,x')+\partial_{x'\nu}F_{\mathrm{V}}^{(\mathrm{f})}{}_{\mu}(x,x')\Big]\right\}\Bigg|_{x'=x}.
\end{aligned}
\tag{B.40}
$$

**Energy Density** We have:

$$
\varepsilon^{(\mathrm{f})}(t) = -4\int_p\Big[m^{(\mathrm{f})}F_{\mathrm{S}}^{(\mathrm{f})}(t,t;p)+p\,F_{\mathrm{V}}^{(\mathrm{f})}(t,t;p)\Big]\,. \tag{B.41}
$$

For a free gas of fermions, we obtain:

$$
\begin{aligned}
\varepsilon_0^{(\mathrm{f})} &= -4\int_p\sqrt{p^2+m^{(\mathrm{f})2}}\left[\frac{1}{2}-n^{(\mathrm{f})}\Big(\sqrt{p^2+m^{(\mathrm{f})2}}\Big)\right] \\
&= -\frac{2}{\pi^2}\int_0^\infty \mathrm{d}p\,p^2\sqrt{p^2+m^{(\mathrm{f})2}}\left[\frac{1}{2}-n^{(\mathrm{f})}\Big(\sqrt{p^2+m^{(\mathrm{f})2}}\Big)\right]\,.
\end{aligned}
\tag{B.42}
$$

The energy density of a free fermion gas is time independent and depends on the fermion distribution $n^{(\mathrm{f})}(p)$. We have:

- Vacuum: $n^{(\mathrm{f})}(p)=0$:

$$
\begin{aligned}
\varepsilon_{0,\mathrm{vac}}^{(\mathrm{f})} &= -\lim_{\Lambda\to\infty}\frac{1}{8\pi^2}\left[\Lambda\sqrt{\Lambda^2+m^{(\mathrm{f})2}}\,(2\Lambda^2+m^{(\mathrm{f})2})+m^{(\mathrm{f})4}\ln\left(\frac{m^{(\mathrm{f})}}{\Lambda+\sqrt{\Lambda^2+m^{(\mathrm{f})2}}}\right)\right] \\
&= -\lim_{\Lambda\to\infty}\frac{\Lambda^4}{4\pi^2}\to-\infty\,.
\end{aligned}
$$

The vacuum energy density does not depend on any parameter and diverges. If, however, a finite momentum cutoff $\Lambda$ is introduced in order to obtain a finite result, it depends on the single parameter $\Lambda$ and resembles a cosmological constant.
For a massless free fermion gas, we obtain:

$$\varepsilon^{(\mathrm{f})}_{0,\mathrm{vac}}\Big|_{m^{(\mathrm{f})}=0} = -\frac{\Lambda^4}{4\pi^2} \,.$$

Note that the fermions contribute *negatively* to the vacuum energy density.

- Thermal equilibrium: $n^{(\mathrm{f})}(p) = n_{\mathrm{FD}}\Big(\sqrt{p^2+m^{(\mathrm{f})2}}\Big) = 1\Big/\Big(e^{\sqrt{p^2+m^{(\mathrm{f})2}}/T}+1\Big)$. The energy density cannot be found analytically in the massive case; for a massless free fermion gas with $n^{(\mathrm{f})}(p) = n_{\mathrm{FD}}(p) = 1/(\mathrm{e}^{p/T}+1)$, we obtain:

$$\varepsilon^{(\mathrm{f})}_{0,\mathrm{therm}}\Big|_{m^{(\mathrm{f})}=0} = \varepsilon^{(\mathrm{f})}_{0,\mathrm{vac}}\Big|_{m^{(\mathrm{f})}=0} + \frac{7\pi^2T^4}{60} \,.$$

Thermal equilibrium depends on a single parameter, the temperature $T$. This is the expected result for a free gas of massless fermions with a degeneracy of four, corresponding to the two spin states, and particle and antiparticle.
Note that the energy density of a free massless QED plasma at temperature $T$ is then given by:

$$\varepsilon^{(\mathrm{g})}_{0,\mathrm{therm}} + \varepsilon^{(\mathrm{f})}_{0,\mathrm{therm}}\Big|_{m^{(\mathrm{f})}=0} = \frac{11\pi^2T^4}{60} - \lim_{\Lambda\to\infty}\frac{\Lambda^4}{8\pi^2} \,.$$

- Gaussian or “tsunami”: $n^{(\mathrm{f})}(p) = n_{\mathrm{Gauss}}(p) = A\exp(-(p-p_0)^2/(2\sigma^2))$. The energy density cannot be found analytically in the massive case; for a massless free fermion gas, we obtain:

$$\begin{aligned}\varepsilon^{(\mathrm{f})}_{0,\mathrm{Gauss}}&\Big|_{m^{(\mathrm{f})}=0}\\ &= \varepsilon^{(\mathrm{f})}_{0,\mathrm{vac}}\Big|_{m^{(\mathrm{f})}=0} + \frac{A\,\sigma}{\pi^2}\left\{\sqrt{2\pi}\,p_0\,(p_0^2+3\sigma^2)\left[1+\mathrm{erf}\left(\frac{p_0}{\sqrt{2}\,\sigma}\right)\right] + 2\sigma\,(p_0^2+2\sigma^2)\,\mathrm{e}^{-p_0^2/(2\sigma^2)}\right\}.\end{aligned}$$

It depends on three parameters, the amplitude $A$, the mean $p_0$, and the width $\sigma$.

- Homogeneous “stream”: $n^{(\mathrm{f})}(p) = n_{\mathrm{stream}}(p) = p_0\,\delta(p-\overline{p})$

$$\varepsilon^{(\mathrm{f})}_{0,\mathrm{stream}} = \varepsilon^{(\mathrm{f})}_{0,\mathrm{vac}} + \frac{2p_0\,\overline{p}\,\sqrt{\overline{p}^2+m^{(\mathrm{f})2}}}{\pi^2} \,.$$

It depends on two parameters, $p_0$ and the stream momentum $p_0$.

**Pressure Density** We have:

$$\begin{aligned}T^{(\mathrm{f})i}{}_i(x) &= -4\left\{-3\Big[\mathrm{i}\,\partial_{x'\mu}F^{(\mathrm{f})}_{\mathrm{V}}(x,x') + m^{(\mathrm{f})}F^{(\mathrm{f})}_{\mathrm{S}}(x,x')\Big] + \mathrm{i}\,\partial_{x'i}F^{(\mathrm{f})i}_{\mathrm{V}}(x,x')\right\}\Bigg|_{x'=x}\\ &= -4\left[\mathrm{i}\,\partial_{x'i}F^{(\mathrm{f})i}_{\mathrm{V}}(x,x') - 3I^{(\mathrm{f})}_{(\mathrm{F})\mathrm{S}}(x,x')\right]\Bigg|_{x'=x} ,\end{aligned} \tag{B.43}$$

where we have inserted the Dirac conjugate fermion EOM[21], so that

$$p^{(\mathrm{f})}(t) = -4 \int_p \Big[ p\, F_{\mathrm{V}}^{(\mathrm{f})}(t,t;p) - 3 I_{(\mathrm{F})\mathrm{S}}^{(\mathrm{f})}(t,t;p) \Big] . \tag{B.44}$$

### B.4.7 Interaction Part

First note that the (two-loop) 2PI part of the effective action can be written equivalently as

$$\Gamma_{\mathrm{2PI}}[S,D] = \frac{\mathrm{i}}{2}\,\mathrm{Tr}\Big(\Sigma S\Big) = \frac{\mathrm{i}}{2}\int_{x,y} \mathrm{tr}\Big(\Sigma(x,y)S(y,x)\Big) = \frac{1}{2}\int_x \mathrm{tr}\Big(I^{(\mathrm{f})}(x,x)\Big) = 2\int_x I_{\mathrm{S}}^{(\mathrm{f})}(x,x) \tag{B.45}$$

or as[22]

$$\Gamma_{\mathrm{2PI}}[S,D] = -\frac{\mathrm{i}}{2}\,\mathrm{Tr}\Big(\Pi D\Big) = -\frac{\mathrm{i}}{2}\int_{x,y} \Pi^{\mu\nu}(x,y) D_{\nu\mu}(y,x) = -\frac{1}{2}\int_x I^{(\mathrm{g})\mu}{}_{\mu}(x,x) . \tag{B.46}$$

Both expressions can in principle be used to derive the interaction part of the energy-momentum tensor. Here we have to be careful with dependencies on the metric, which are (apart from the integral measures) "hidden" in the self-energies (i. e. are not explicit in the above expressions). Further, gamma matrices, which also depend in a nontrivial way on the metric, are hidden in the self-energies as well.

[21]The fermion EOM for the statistical function reads:

$$\big(\mathrm{i}\,\gamma^{\mu}\partial_{x\mu} - m^{(\mathrm{f})}\big) F^{(\mathrm{f})}(x,y) = I_{(\mathrm{F})}^{(\mathrm{f})}(x,y) .$$

It follows by hermitean conjugation:

$$\begin{aligned}
&\Big[\big(\mathrm{i}\,\gamma^{\mu}\partial_{x\mu} - m^{(\mathrm{f})}\big) F^{(\mathrm{f})}(x,y)\Big]^{\dagger} = I_{(\mathrm{F})}^{(\mathrm{f})}(x,y)^{\dagger} \\
\Leftrightarrow &-\mathrm{i}\,\partial_{x\mu} F^{(\mathrm{f})}(x,y)^{\dagger}\gamma^{\mu\dagger} - m^{(\mathrm{f})} F^{(\mathrm{f})}(x,y)^{\dagger} = I_{(\mathrm{F})}^{(\mathrm{f})}(x,y)^{\dagger} \\
\Leftrightarrow &-\mathrm{i}\,\gamma^{0}\partial_{x\mu} F^{(\mathrm{f})}(y,x)\gamma^{0}\gamma^{0}\gamma^{\mu}\gamma^{0} - m^{(\mathrm{f})}\gamma^{0} F^{(\mathrm{f})}(y,x)\gamma^{0} = I_{(\mathrm{F})}^{(\mathrm{f})}(x,y)^{\dagger} \\
\Leftrightarrow &-\Big[\mathrm{i}\,\partial_{x\mu} F^{(\mathrm{f})}(y,x)\gamma^{\mu} + m^{(\mathrm{f})} F^{(\mathrm{f})}(y,x)\Big] = \gamma^{0} I_{(\mathrm{F})}^{(\mathrm{f})}(x,y)^{\dagger}\gamma^{0} \\
\Leftrightarrow &-\Big[\mathrm{i}\,\partial_{y\mu} F^{(\mathrm{f})}(x,y)\gamma^{\mu} + m^{(\mathrm{f})} F^{(\mathrm{f})}(x,y)\Big] = \gamma^{0} I_{(\mathrm{F})}^{(\mathrm{f})}(y,x)^{\dagger}\gamma^{0}
\end{aligned}$$

where in the last step we have interchanged the arguments. Taking the trace of this equation, we obtain:

$$\mathrm{i}\,\partial_{y\mu} F_{\mathrm{V}}^{(\mathrm{f})\mu}(x,y) + m^{(\mathrm{f})} F_{\mathrm{S}}^{(\mathrm{f})}(x,y) = -I_{(\mathrm{F})\mathrm{S}}^{(\mathrm{f})}(y,x) ,$$

where we have used that the scalar component of the memory integral is real. It finally follows that

$$\Big[\mathrm{i}\,\partial_{y\mu} F_{\mathrm{V}}^{(\mathrm{f})\mu}(x,y) + m^{(\mathrm{f})} F_{\mathrm{S}}^{(\mathrm{f})}(x,y)\Big]\Big|_{y=x} = -I_{(\mathrm{F})\mathrm{S}}^{(\mathrm{f})}(x,y)\big|_{y=x} .$$

[22]Note that $\Pi^{\mu\nu}(x,y)D_{\nu\mu}(y,x) = \Pi^{\mu\nu}(x,y)D_{\mu\nu}(x,y)$.

We will first show that both expressions yield the same result. In order to do so, we have to show that

$$\mathrm{tr}\left(\frac{\delta\Sigma(y,x)}{\delta g^{\mu\nu}(x)}\,S(y,x)\right) = -\frac{\delta\Pi^{\rho\sigma}(y,z)}{\delta g^{\mu\nu}(x)}\,D_{\sigma\rho}(z,y)\,.$$

With

$$\Pi^{\rho\sigma}(y,z) = e^2\,\mathrm{tr}\Big(\gamma^\rho(y)S(y,z)\gamma^\sigma(z)S(z,y)\Big)\,,$$

we have:

$$\begin{aligned}
&\frac{\delta\Pi^{\rho\sigma}(y,z)}{\delta g^{\mu\nu}(x)}\\
&\quad = \frac{e^2}{4}\,\mathrm{tr}\bigg(\quad\Big[\delta^\rho_\mu\gamma_\nu(y)+\delta^\rho_\nu\gamma_\mu(y)\Big]S(y,z)\gamma^\sigma(z)S(z,y)\delta^4(y-x)\\
&\qquad\qquad + \gamma^\rho(y)S(y,z)\Big[\delta^\sigma_\mu\gamma_\nu(z)+\delta^\sigma_\nu\gamma_\mu(z)\Big]S(z,y)\delta^4(z-x)\bigg)\\
&\quad = \frac{1}{4}\bigg\{\Big[\delta^\rho_\mu\Pi_\nu{}^\sigma(y,z)+\delta^\rho_\nu\Pi_\mu{}^\sigma(y,z)\Big]\delta^4(y-x)+\Big[\delta^\sigma_\mu\Pi^\rho{}_\nu(y,z)+\delta^\sigma_\nu\Pi^\rho{}_\mu(y,z)\Big]\delta^4(z-x)\bigg\}\,,
\end{aligned}$$

so that

$$\begin{aligned}
\frac{\delta\Pi^{\rho\sigma}(y,z)}{\delta g^{\mu\nu}(x)}\,D_{\sigma\rho}(y,z) = \frac{1}{4}\bigg\{&\quad\Big[\Pi_\nu{}^\rho(y,z)D_{\rho\mu}(z,y)+\Pi_\mu{}^\rho(y,z)D_{\rho\nu}(z,y)\Big]\delta^4(y-x)\\
&+\Big[\Pi^\rho{}_\nu(y,z)D_{\mu\rho}(z,y)+\Pi^\rho{}_\mu(y,z)D_{\nu\rho}(z,y)\Big]\delta^4(z-x)\bigg\}\,,
\end{aligned}$$

and

$$\begin{aligned}
-\frac{\delta\Sigma(y,z)}{\delta g^{\mu\nu}(x)} = \frac{e^2}{4}\bigg\{&\quad\Big[\delta^\rho_\mu\gamma_\nu(y)+\delta^\rho_\nu\gamma_\mu(y)\Big]\delta^4(y-x)S(y,z)\gamma^\sigma(z)D_{\rho\sigma}(y,z)\\
&+\gamma^\rho(y)S(y,z)\Big[\delta^\sigma_\mu\gamma_\nu(z)+\delta^\sigma_\nu\gamma_\mu(z)\Big]\delta^4(z-x)D_{\rho\sigma}(y,z)\bigg\}\,,
\end{aligned}$$

so that

$$\begin{aligned}
&-\mathrm{tr}\left(\frac{\delta\Sigma(y,z)}{\delta g^{\mu\nu}(x)}\,S(z,y)\right)\\
&\quad = \frac{e^2}{4}\bigg\{\quad\Big[\delta^\rho_\mu\gamma_\nu(y)+\delta^\rho_\nu\gamma_\mu(y)\Big]S(y,z)\gamma^\sigma(z)S(z,y)D_{\rho\sigma}(y,z)\delta^4(y-x)\\
&\qquad\qquad + \gamma^\rho(y)S(y,z)\Big[\delta^\sigma_\mu\gamma_\nu(z)+\delta^\sigma_\nu\gamma_\mu(z)\Big]S(z,y)D_{\rho\sigma}(y,z)\delta^4(z-x)\bigg\}\\
&\quad = \frac{1}{4}\bigg\{\quad\Big[\Pi_\nu{}^\rho(y,z)D_{\mu\rho}(y,z)+\Pi_\mu{}^\rho(y,z)D_{\nu\rho}(y,z)\Big]\delta^4(y-x)\\
&\qquad\qquad + \Big[\Pi^\rho{}_\nu(y,z)D_{\rho\mu}(y,z)+\Pi^\rho{}_\mu(y,z)D_{\rho\nu}(y,z)\Big]\delta^4(z-x)\bigg\}\,,
\end{aligned}$$

i. e. both expressions yield the same interaction energy-momentum tensor.

Expressing the interaction part through the photon propagator and self-energy, we have:

$$\Gamma_{2\mathrm{PI}}[S,D,g] = -\frac{\mathrm{i}}{2}\int_{y,z}\sqrt{-g(y)}\,\sqrt{-g(z)}\,\Pi^{\rho\sigma}(y,z)D_{\sigma\rho}(z,y)\,.$$

We then obtain:

$$\begin{aligned}
&\frac{\delta\Gamma_{2\mathrm{PI}}[S,D,g]}{\delta g^{\mu\nu}(x)}\\
&\quad= -\frac{\mathrm{i}}{2}\int_{y,z}\sqrt{-g(y)}\,\sqrt{-g(z)}\\
&\qquad\qquad\cdot\left\{-\frac{1}{2}\Big[g_{\mu\nu}(y)\delta^4(y-x)+g_{\mu\nu}(z)\delta^4(z-x)\Big]\Pi^{\rho\sigma}(y,z)D_{\sigma\rho}(z,y)\right.\\
&\qquad\qquad\quad\left.+\frac{\delta\Pi^{\rho\sigma}(y,z)}{\delta g^{\mu\nu}(x)}\,D_{\sigma\rho}(z,y)\right\}\\
&\quad= -\frac{\mathrm{i}}{2}\int_{y,z}\sqrt{-g(y)}\,\sqrt{-g(z)}\,\delta^4(y-x)\\
&\qquad\qquad\cdot\left\{-\frac{1}{2}\,g_{\mu\nu}(y)\Pi^{\rho\sigma}(y,z)D_{\sigma\rho}(z,y)+\frac{1}{4}\Big[\Pi_\nu{}^\rho(y,z)D_{\rho\mu}(z,y)+\Pi_\mu{}^\rho(y,z)D_{\rho\nu}(z,y)\Big]\right\}\\
&\qquad-\frac{\mathrm{i}}{2}\int_{y,z}\sqrt{-g(y)}\,\sqrt{-g(z)}\,\delta^4(z-x)\\
&\qquad\qquad\cdot\left\{-\frac{1}{2}\,g_{\mu\nu}(z)\Pi^{\rho\sigma}(y,z)D_{\sigma\rho}(z,y)+\frac{1}{4}\Big[\Pi^\rho{}_\nu(y,z)D_{\mu\rho}(z,y)+\Pi^\rho{}_\mu(y,z)D_{\nu\rho}(z,y)\Big]\right\}\\
&\quad= \quad\frac{\mathrm{i}}{4}\int_z\sqrt{-g(z)}\left\{g_{\mu\nu}(x)\Pi^{\rho\sigma}(x,z)D_{\sigma\rho}(z,x)-\frac{1}{2}\Big[\Pi_\nu{}^\rho(x,z)D_{\rho\mu}(z,x)+\Pi_\mu{}^\rho(x,z)D_{\rho\nu}(z,x)\right\}\\
&\qquad+\frac{\mathrm{i}}{4}\int_y\sqrt{-g(y)}\left\{g_{\mu\nu}(x)\Pi^{\rho\sigma}(y,x)D_{\sigma\rho}(x,y)-\frac{1}{2}\Big[\Pi^\rho{}_\nu(y,x)D_{\mu\rho}(x,y)+\Pi^\rho{}_\mu(y,x)D_{\nu\rho}(x,y)\right\}\\
&\quad= \frac{\mathrm{i}}{2}\int_y\sqrt{-g(y)}\left\{g_{\mu\nu}(x)\Pi^{\rho\sigma}(x,y)D_{\sigma\rho}(y,x)-\frac{1}{2}\Big[\Pi_\mu{}^\rho(x,y)D_{\rho\nu}(y,x)+\Pi_\nu{}^\rho(x,y)D_{\rho\mu}(y,x)\right\}\\
&\quad= \frac{1}{2}\left\{g_{\mu\nu}(x)I^{(\mathrm{g})\rho}{}_\rho(x,x)-\frac{1}{2}\Big[I^{(\mathrm{g})}_{\mu\nu}(x,x)+I^{(\mathrm{g})}_{\nu\mu}(x,x)\Big]\right\},
\end{aligned}$$

so that

$$T^{(\mathrm{int})}_{\mu\nu}(x) = 2\,\frac{\delta\Gamma_{2\mathrm{PI}}[S,D,g]}{\delta g^{\mu\nu}(x)}\bigg|_{g_{\mu\nu}(x)=\eta_{\mu\nu}} = \eta_{\mu\nu}I^{(\mathrm{g})\rho}_{(\mathrm{F})\ \rho}(x,x)-\frac{1}{2}\Big[I^{(\mathrm{g})}_{(\mathrm{F})\mu\nu}(x,x)+I^{(\mathrm{g})}_{(\mathrm{F})\nu\mu}(x,x)\Big] \quad \text{(B.47)}$$

with

$$I^{(\mathrm{g})}_{(\mathrm{F})\mu\nu}(x,x) = \int \mathrm{d}^3y\int_0^{x^0}\mathrm{d}y^0\Big[\Pi_{(\rho)\mu}{}^\rho(x,y)F^{(\mathrm{g})}_{\rho\nu}(y,x)-\Pi_{(F)\mu}{}^\rho(x,y)\rho^{(\mathrm{g})}_{\rho\nu}(y,x)\Big]. \quad \text{(B.48)}$$

Note that

$$T^{(\mathrm{int})\mu}{}_\mu(x) = 3I^{(\mathrm{g})\mu}_{(\mathrm{F})\ \mu}(x,x) = -12I^{(\mathrm{f})}_{\mathrm{S}}(x,x)\,,$$

i. e. the interaction is conformal if the scalar component of the statistical part of the fermion memory integral vanishes, which is the case for massless fermions.

**Energy Density** The interaction energy density is then given by:

$$\varepsilon^{(\mathrm{int})}(t) = -\int_p \left[ 2I^{(\mathrm{g})}_{(\mathrm{F})\mathrm{T}}(t,t;p) + I^{(\mathrm{g})}_{(\mathrm{F})\mathrm{L}}(t,t;p) \right] . \tag{B.49}$$

**Pressure Density** The pressure density reads:

$$p^{(\mathrm{int})}(t) = \int_p \left\{ 3I^{(\mathrm{g})}_{(\mathrm{F})\mathrm{S}}(t,t;p) - 2\left[ 2I^{(\mathrm{g})}_{(\mathrm{F})\mathrm{T}}(t,t;p) + I^{(\mathrm{g})}_{(\mathrm{F})\mathrm{L}}(t,t;p) \right] \right\} . \tag{B.50}$$

# Appendix C

# Generalized Convolutions

In this appendix, we will define *generalized convolutions*. In contrast to the standard convolution, the generalized convolutions include certain angle-dependent prefactors which appear in the convolutions due to the tensor structure of the propagators entering the self-energies. It is not necessary to define such generalized convolutions, but turns out be very practical as an abstraction.

The importance lies in fact more in the possibility to accordingly define *generalized convolution theorems* in a manner similar to the convolution theorem for the standard convolution. In spite of the possibly complicated tensor structure of the integrands in the self-energies, this then allows for a much simpler expression of the self-energies in terms of generalized convolutions.

In order to distinguish the convolution as it is commonly known from the generalized convolutions we are about to define, we will usually denote the known convolution as *standard convolution*.

### The Standard Convolution

We are usually interested in functions whose arguments are spatial vectors, and integrals over these functions are over the volume. If the functions are scalar-valued, they may only depend on the modulus of the spatial momentum due to the assumption of spatial isotropy.

Given two such functions $f_1$ and $f_2$, the (standard) convolution $g = f_1 * f_2$ is defined as

$$g(|\boldsymbol{p}|) = (f_1 * f_2)(|\boldsymbol{p}|) = \int_{\boldsymbol{q}} f_1(|\boldsymbol{q}|)\, f_2(|\boldsymbol{p}-\boldsymbol{q}|)\,. \tag{C.1}$$

Note that, in contrast to the convolution of two functions defined on the real line (i. e. which are one-dimensional), there is an angular dependency in our case in the argument of $f_2$. In fact, one has

$$\begin{aligned}\int_{\boldsymbol{q}} f_1(|\boldsymbol{q}|) f_2(|\boldsymbol{p}-\boldsymbol{q}|) &= \frac{1}{4\pi^2}\int_0^\infty \mathrm{d}q\, q^2 f_1(q) \int_{-1}^{1} \mathrm{d}x\, f_2\Big(\sqrt{p^2+q^2-2\,p\,q\,x}\Big) \\ &= \frac{1}{4\pi^2\,p}\int_0^\infty \mathrm{d}q\, q\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, k\, f_2(k)\end{aligned} \tag{C.2}$$

with $x = \cos(\theta) = \boldsymbol{p}\cdot\boldsymbol{q}/(|\boldsymbol{p}||\boldsymbol{q}|) \in [-1;1]$, where $\theta$ is the angle formed by $\boldsymbol{p}$ and $\boldsymbol{q}$, and $k = \sqrt{p^2+q^2-2\,p\,q\,x} = |\boldsymbol{p}-\boldsymbol{q}| \in [|p-q|;p+q] \subset [0,\infty]$.[1]

### The Standard Convolution Theorem

Given two isotropic functions $f_1$ and $f_2$, their convolution $g = f_1 * f_2$ is given by:

$$\begin{aligned} g(|\boldsymbol{p}|) &= (f_1 * f_2)(|\boldsymbol{p}|) \\ &= \int_{\boldsymbol{q}} f_1(|\boldsymbol{q}|)\, f_2(|\boldsymbol{p}-\boldsymbol{q}|) \\ &= \int_{\boldsymbol{q}} \int_{\boldsymbol{k}} f_1(|\boldsymbol{q}|)\, f_2(|\boldsymbol{k}|)\, \delta^3\Big(\boldsymbol{k} - (\boldsymbol{p}-\boldsymbol{q})\Big) \\ &= \int_{\boldsymbol{x}} \int_{\boldsymbol{q}} \int_{\boldsymbol{k}} f_1(|\boldsymbol{q}|)\, f_2(|\boldsymbol{k}|)\, \mathrm{e}^{-\mathrm{i}\,\boldsymbol{x}\cdot[\boldsymbol{k}-(\boldsymbol{p}-\boldsymbol{q})]} \\ &= \int_{\boldsymbol{x}} \left[\int_{\boldsymbol{q}} f_1(|\boldsymbol{q}|)\, \mathrm{e}^{-\mathrm{i}\,\boldsymbol{q}\cdot\boldsymbol{x}}\right] \left[\int_{\boldsymbol{k}} f_2(|\boldsymbol{k}|)\, \mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\right] \mathrm{e}^{\mathrm{i}\,\boldsymbol{p}\cdot\boldsymbol{x}}\ . \end{aligned} \tag{C.3}$$

This is the standard convolution theorem: The Fourier transform of the convolution of two functions is equal to the product of the Fourier transforms of the two functions. This is the only kind of convolution which appears in the self-energies of scalar theories. In theories with a nontrivial spatial tensor structure like gauge theories, however, one encounters more complicated types of convolutions which we will discuss in the following.

## C.1 Generalized Convolution Theorems

### Generalized Convolutions

A generalized convolution of type A, $g$, of two scalar functions $f_1$ and $f_2$ is of the form

$$\begin{aligned} g(|\boldsymbol{p}|) &= (f_1 *_{\mathrm{A}} f_2)(|\boldsymbol{p}|) \\ &= \int_{\boldsymbol{q}} \alpha_{\mathrm{A}}(|\boldsymbol{p}|,|\boldsymbol{q}|,|\boldsymbol{p}-\boldsymbol{q}|)\, f_1(|\boldsymbol{q}|) f_2(|\boldsymbol{p}-\boldsymbol{q}|) \\ &= \frac{1}{4\pi^2}\int_0^\infty \mathrm{d}q\, q^2 f_1(q) \int_{-1}^{1} \mathrm{d}x\, \alpha_{\mathrm{S}}\Big(p,q,\sqrt{p^2+q^2-2\,p\,q\,x}\Big)\, f_2\Big(\sqrt{p^2+q^2-2\,p\,q\,x}\Big) \\ &= \frac{1}{4\pi^2\,p}\int_0^\infty \mathrm{d}q\, q\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, k\, \alpha_{\mathrm{S}}(p,q,k)\, f_2(k)\,, \end{aligned} \tag{C.4}$$

where $\alpha_{\mathrm{A}}$ is a prefactor which depends in particular on the angle between the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$. Note that the standard convolution is obtained in the special case that this function is unity (and hence in particular independent of the angle). Note that, since the integrand is still a scalar, the generalized convolution depends only on the modulus of $\boldsymbol{p}$, just as for the standard convolution.

[1]For a finite spatial momentum cutoff $\Lambda$, one has $p,q \in [0,\Lambda]$, so that $[|p-q|;p+q] \subset [0,2\,\Lambda]$.

The origin of such prefactors is the tensor structure of the integrand functions. For instance, for two vector-valued functions $f_1^i$ and $f_2^i$, one can consider the convolution of their contraction, i. e.

$$\begin{aligned}\int_{\boldsymbol{q}} f_{1i}(\boldsymbol{q}) f_2^i(\boldsymbol{p}-\boldsymbol{q}) &= \int_{\boldsymbol{q}} \frac{q_i}{|\boldsymbol{q}|}\, f_1(|\boldsymbol{q}|)\, \frac{p^i - q^i}{|\boldsymbol{p}-\boldsymbol{q}|}\, f_2(|\boldsymbol{p}-\boldsymbol{q}|) \\ &= -\int_{\boldsymbol{q}} \frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\, f_1(|\boldsymbol{q}|) f_2(|\boldsymbol{p}-\boldsymbol{q}|)\,. \end{aligned} \tag{C.5}$$

### Generalized Convolution Theorems

The problem remains of how to cope with the angle-dependent prefactors. In the end, we would like to formulate generalized convolution theorems in analogy to the standard convolution theorem. Since in the numerical implementation, however, we work on an isotropic grid (see App. E), it is not clear what to do with the angle-dependent prefactors since on our grid there is no notion of “angle”. Fortunately, however, it turns out that one can do without ever making use of the angle if one is only interested in the convolution.[2]

**Rewriting Prefactors with Angular Dependencies** Our goal is to bring the integral which defines the respective generalized convolution into a form in which there is no longer an explicit dependence on the angle. This can be achieved by employing polarization identities in order to rewrite the angle-dependent prefactors. We have, for instance,

$$\boldsymbol{p}\cdot\boldsymbol{q} = \frac{1}{2}\boldsymbol{p}^2 + \frac{1}{2}\boldsymbol{q}^2 - \frac{1}{2}(\boldsymbol{p}-\boldsymbol{q})^2\,.$$

It immediately follows that

$$\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q}) = \frac{1}{2}(\boldsymbol{p}-\boldsymbol{q})^2 + \frac{1}{2}\boldsymbol{p}^2 - \frac{1}{2}\boldsymbol{q}^2\,, \quad \boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q}) = \frac{1}{2}\boldsymbol{p}^2 - \frac{1}{2}\boldsymbol{q}^2 - \frac{1}{2}(\boldsymbol{p}-\boldsymbol{q})^2\,,$$

so that

$$\begin{aligned}(\boldsymbol{p}\cdot\boldsymbol{q})\,\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q}) &= \frac{1}{4}\Big[\boldsymbol{p}^2 + \boldsymbol{q}^2 - (\boldsymbol{p}-\boldsymbol{q})^2\Big]\Big[(\boldsymbol{p}-\boldsymbol{q})^2 + \boldsymbol{p}^2 - \boldsymbol{q}^2\Big] \\ &= \frac{1}{4}\Big\{\boldsymbol{p}^2 + \Big[\boldsymbol{q}^2 - (\boldsymbol{p}-\boldsymbol{q})^2\Big]\Big\}\Big\{\boldsymbol{p}^2 - \Big[\boldsymbol{q}^2 - (\boldsymbol{p}-\boldsymbol{q})^2\Big]\Big\} \\ &= \frac{1}{4}\Big\{\boldsymbol{p}^4 - \Big[\boldsymbol{q}^2 - (\boldsymbol{p}-\boldsymbol{q})^2\Big]^2\Big\} \\ &= \frac{1}{4}\Big[\boldsymbol{p}^4 - \boldsymbol{q}^4 + 2\,\boldsymbol{q}^2(\boldsymbol{p}-\boldsymbol{q})^2 - (\boldsymbol{p}-\boldsymbol{q})^4\Big]\,.\end{aligned}$$

[2]Actually, it *must* be possible to do the necessary calculations without resorting to angles due to our assumption of isotropy.

For instance, for a vector-valued function $f_1^i$ and a scalar-valued function $f_2$, we then have:

$$
\begin{aligned}
g_1(|\boldsymbol{p}|) &= -\frac{p_i}{|\boldsymbol{p}|}\,(f_1^i * f_2)(|\boldsymbol{p}|) \\
&= -\frac{p_i}{|\boldsymbol{p}|}\int_{\boldsymbol{q}} f_1^i(\boldsymbol{q})\, f_2(\boldsymbol{p}-\boldsymbol{q}) \\
&= \int_{\boldsymbol{q}} \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\, f_{1\mathrm{V}}(|\boldsymbol{q}|) f_{2\mathrm{S}}(|\boldsymbol{p}-\boldsymbol{q}|) \\
&= -\frac{1}{2}\int_{\boldsymbol{q}} \frac{(\boldsymbol{p}-\boldsymbol{q})^2-\boldsymbol{p}^2-\boldsymbol{q}^2}{|\boldsymbol{p}||\boldsymbol{q}|}\, f_{1\mathrm{V}}(|\boldsymbol{q}|)\, f_{2\mathrm{S}}(|\boldsymbol{p}-\boldsymbol{q}|) \\
&= -\frac{1}{2}\int_{\boldsymbol{q}}\int_{\boldsymbol{k}} \frac{\boldsymbol{k}^2-\boldsymbol{p}^2-\boldsymbol{q}^2}{|\boldsymbol{p}||\boldsymbol{q}|}\, f_{1\mathrm{V}}(|\boldsymbol{q}|)\, f_{2\mathrm{S}}(|\boldsymbol{k}|)\, \delta^3\big(\boldsymbol{k}-(\boldsymbol{p}-\boldsymbol{q})\big) \\
&= -\frac{1}{2}\int_{\boldsymbol{x}}\int_{\boldsymbol{q}}\int_{\boldsymbol{k}} \frac{\boldsymbol{k}^2-\boldsymbol{p}^2-\boldsymbol{q}^2}{|\boldsymbol{p}||\boldsymbol{q}|}\, f_{1\mathrm{V}}(|\boldsymbol{q}|)\, f_{2\mathrm{S}}(|\boldsymbol{k}|)\, \mathrm{e}^{-\mathrm{i}\,\boldsymbol{x}\cdot(\boldsymbol{k}-\boldsymbol{p}+\boldsymbol{q})} \\
&= \frac{1}{2}\int_{\boldsymbol{x}}\Bigg\{\ |\boldsymbol{p}|\left[\int_{\boldsymbol{q}}\frac{1}{|\boldsymbol{q}|}\, f_{1\mathrm{V}}(|\boldsymbol{q}|)\,\mathrm{e}^{-\mathrm{i}\,\boldsymbol{q}\cdot\boldsymbol{x}}\right]\left[\int_{\boldsymbol{k}} f_{2\mathrm{S}}(|\boldsymbol{k}|)\,\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\right] \\
&\qquad\quad + \frac{1}{|\boldsymbol{p}|}\left[\int_{\boldsymbol{q}}|\boldsymbol{q}|\, f_{1\mathrm{V}}(|\boldsymbol{q}|)\,\mathrm{e}^{-\mathrm{i}\,\boldsymbol{q}\cdot\boldsymbol{x}}\right]\left[\int_{\boldsymbol{k}} f_{2\mathrm{S}}(|\boldsymbol{k}|)\,\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\right] \\
&\qquad\quad - \frac{1}{|\boldsymbol{p}|}\left[\int_{\boldsymbol{q}}\frac{1}{|\boldsymbol{q}|}\, f_{1\mathrm{V}}(|\boldsymbol{q}|)\,\mathrm{e}^{-\mathrm{i}\,\boldsymbol{q}\cdot\boldsymbol{x}}\right]\left[\int_{\boldsymbol{k}}|\boldsymbol{k}|^2\, f_{2\mathrm{S}}(|\boldsymbol{k}|)\,\mathrm{e}^{-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{x}}\right]\Bigg\}\,\mathrm{e}^{\mathrm{i}\,\boldsymbol{p}\cdot\boldsymbol{x}}\ .
\end{aligned}
\tag{C.6}
$$

Similarly, one obtains:

$$
g_2(|\boldsymbol{p}|) = -\frac{p_i}{|\boldsymbol{p}|}\,(f_2 * f_1^i)(|\boldsymbol{p}|) = \int_{\boldsymbol{q}}\frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\, f_{2\mathrm{S}}(|\boldsymbol{q}|) f_{1\mathrm{V}}(|\boldsymbol{p}-\boldsymbol{q}|) = \ldots = g_1(|\boldsymbol{p}|) \tag{C.7}
$$

(where we left out some steps which work in complete analogy to the previous calculation), since the convolution is commutative. Note that the angular dependence is effectively integrated out.

The following structures (which are all scalars under spatial rotations) occur:

$$
\begin{aligned}
f(\boldsymbol{q})\, g(\boldsymbol{p}-\boldsymbol{q}) &= \alpha_{\mathrm{SS}}(|\boldsymbol{p}|,|\boldsymbol{q}|,|\boldsymbol{p}-\boldsymbol{q}|)\, f_{\mathrm{S}}(|\boldsymbol{q}|)\, g_{\mathrm{S}}(|\boldsymbol{p}-\boldsymbol{q}|)\,, \\
f_i(\boldsymbol{q})\, g^i(\boldsymbol{p}-\boldsymbol{q}) &= -\alpha_{\mathrm{VV}_1}(|\boldsymbol{p}|,|\boldsymbol{q}|,|\boldsymbol{p}-\boldsymbol{q}|)\, f_{\mathrm{V}}(|\boldsymbol{q}|)\, g_{\mathrm{V}}(|\boldsymbol{p}-\boldsymbol{q}|)\,, \\
\frac{p_i}{|\boldsymbol{p}|}\, f^i(\boldsymbol{q})\, g(\boldsymbol{p}-\boldsymbol{q}) &= -\alpha_{\mathrm{VS}}(|\boldsymbol{p}|,|\boldsymbol{q}|,|\boldsymbol{p}-\boldsymbol{q}|)\, f_{\mathrm{V}}(|\boldsymbol{q}|)\, g_{\mathrm{S}}(|\boldsymbol{p}-\boldsymbol{q}|)\,, \\
\frac{p_i}{|\boldsymbol{p}|}\, f(\boldsymbol{q})\, g^i(\boldsymbol{p}-\boldsymbol{q}) &= -\alpha_{\mathrm{SV}}(|\boldsymbol{p}|,|\boldsymbol{q}|,|\boldsymbol{p}-\boldsymbol{q}|)\, f_{\mathrm{S}}(|\boldsymbol{q}|)\, g_{\mathrm{V}}(|\boldsymbol{p}-\boldsymbol{q}|)\,, \\
\frac{p_i}{|\boldsymbol{p}|}\, f^{ij}(\boldsymbol{q})\, g_j(\boldsymbol{p}-\boldsymbol{q}) &= \alpha_{\mathrm{TV}}(|\boldsymbol{p}|,|\boldsymbol{q}|,|\boldsymbol{p}-\boldsymbol{q}|)\, f_{\mathrm{L}}(|\boldsymbol{q}|)\, g_{\mathrm{V}}(|\boldsymbol{p}-\boldsymbol{q}|)\,, \\
\frac{p_i}{|\boldsymbol{p}|}\, f_j(\boldsymbol{q})\, g^{ji}(\boldsymbol{p}-\boldsymbol{q}) &= \alpha_{\mathrm{VT}}(|\boldsymbol{p}|,|\boldsymbol{q}|,|\boldsymbol{p}-\boldsymbol{q}|)\, f_{\mathrm{V}}(|\boldsymbol{q}|)\, g_{\mathrm{L}}(|\boldsymbol{p}-\boldsymbol{q}|)\,,
\end{aligned}
$$

$$\frac{p_i\, p_j}{\boldsymbol{p}^2}\, f^i(\boldsymbol{q})\, g^j(\boldsymbol{p}-\boldsymbol{q}) = \alpha_{\mathrm{VV}_2}(|\boldsymbol{p}|, |\boldsymbol{q}|, |\boldsymbol{p}-\boldsymbol{q}|)\, f_{\mathrm{V}}(|\boldsymbol{q}|)\, g_{\mathrm{V}}(|\boldsymbol{p}-\boldsymbol{q}|)$$

with the angular-dependent prefactors

$$\alpha_{\mathrm{SS}}(|\boldsymbol{p}|, |\boldsymbol{q}|, |\boldsymbol{p}-\boldsymbol{q}|) = 1\,, \tag{C.8a}$$

$$\alpha_{\mathrm{VV}_1}(|\boldsymbol{p}|, |\boldsymbol{q}|, |\boldsymbol{p}-\boldsymbol{q}|) = \frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,, \tag{C.8b}$$

$$\alpha_{\mathrm{VS}}(|\boldsymbol{p}|, |\boldsymbol{q}|, |\boldsymbol{p}-\boldsymbol{q}|) = \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,, \tag{C.8c}$$

$$\alpha_{\mathrm{SV}}(|\boldsymbol{p}|, |\boldsymbol{q}|, |\boldsymbol{p}-\boldsymbol{q}|) = \frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,, \tag{C.8d}$$

$$\alpha_{\mathrm{TV}}(|\boldsymbol{p}|, |\boldsymbol{q}|, |\boldsymbol{p}-\boldsymbol{q}|) = \frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|}\,\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,, \tag{C.8e}$$

$$\alpha_{\mathrm{VT}}(|\boldsymbol{p}|, |\boldsymbol{q}|, |\boldsymbol{p}-\boldsymbol{q}|) = \frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\frac{\boldsymbol{q}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{q}||\boldsymbol{p}-\boldsymbol{q}|}\,, \tag{C.8f}$$

$$\alpha_{\mathrm{VV}_2}(|\boldsymbol{p}|, |\boldsymbol{q}|, |\boldsymbol{p}-\boldsymbol{q}|) = \frac{\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{q})}{|\boldsymbol{p}||\boldsymbol{p}-\boldsymbol{q}|}\,\frac{\boldsymbol{p}\cdot\boldsymbol{q}}{|\boldsymbol{p}||\boldsymbol{q}|} \tag{C.8g}$$

determining the type of generalized convolution. We will refer to them respectively as the *scalar-scalar (standard)*, *type 1 vector-vector*, *vector-scalar*, *scalar-vector*, *tensor-vector*, *vector-tensor*, and *type 2 vector-vector* convolutions.[3]

Note that there is no contraction of two tensorial quantities (like $f_{ij}(\boldsymbol{q})g^{ij}(\boldsymbol{p}-\boldsymbol{q})$ or $f_{ik}(\boldsymbol{q})g^{kj}(\boldsymbol{p}-\boldsymbol{q})$) since only the photon propagator contains a tensorial structure. However, there are no products of two or more photon propagators in the one-loop self-energies.

By applying the polarization identities mentioned above, we obtain the following formula for the generalized convolution of type $\mathrm{A} \in \{\mathrm{SS}, \mathrm{VV}_1, \mathrm{VS}, \mathrm{SV}, \mathrm{TV}, \mathrm{VT}, \mathrm{VV}_2\}$:

$$f *_{\mathrm{A}} g := \sum_{(\alpha,(\ell,m,n))\in\mathcal{A}} \alpha\, \mathrm{id}^{\ell}\, \mathcal{F}\Big(\mathcal{F}^{-1}(\mathrm{id}^m f)\, \mathcal{F}^{-1}(\mathrm{id}^n g)\Big) \tag{C.9}$$

with the identity function id (so that $\mathrm{id}(p) = p$), $(\alpha, (\ell, m, n)) \in \mathbb{Q}\times\mathbb{Z}^3$, and $\mathcal{A} \in \{\mathcal{SS}, \mathcal{VV}_1, \mathcal{VS}, \mathcal{SV}, \mathcal{TV}, \mathcal{VT}, \mathcal{VV}_2\}$ with

$$\begin{aligned}
\mathcal{SS} &= \Big\{(1,(0,0,0))\Big\}\,,\\
\mathcal{VV}_1 &= \Big\{(1/2,(2,-1,-1)), (-1/2,(0,1,-1)), (-1/2,(0,-1,1))\Big\}\,,\\
\mathcal{VS} &= \Big\{(1/2,(1,-1,0)), (1/2,(-1,1,0)), (-1/2,(-1,-1,2))\Big\}\,,\\
\mathcal{SV} &= \Big\{(1/2,(1,0,-1)), (1/2,(-1,0,1)), (-1/2,(-1,2,-1))\Big\}\,,
\end{aligned}$$

[3] The nomenclature of the different types of generalized convolutions is chosen such that the tensorial structure of the two quantities which are convolved is indicated, as is easily seen in Eqs. (C.8). For instance, in the vector-scalar convolution, one of the arguments is a vectorial quantity, while the other one is a scalar quantity.

$$\mathcal{TV} = \Big\{(1/4,(3,-2,-1)),(-1/4,(-1,2,-1)),(1/4,(-1,-2,3)),(-1/2,(1,-2,1))\Big\},$$
$$\mathcal{VT} = \Big\{(1/4,(3,-1,-2)),(-1/4,(-1,-1,2)),(1/4,(-1,3,-2)),(-1/2,(1,1,-2))\Big\},$$
$$\mathcal{VV}_2 = \Big\{(1/4,(2,-1,-1)),(-1/4,(-2,3,-1)),(1/2,(-2,1,1)),(-1/4,(-2,-1,3))\Big\}.$$

Explicitly, the different types of generalized convolutions read:

$$f *_{\mathrm{SS}} g = \mathcal{F}\Big(\mathcal{F}^{-1}(f)\,\mathcal{F}^{-1}(g)\Big) = g *_{\mathrm{SS}} f = f * g\,, \tag{C.10a}$$

$$\begin{aligned} f *_{\mathrm{VV}_1} g &= \frac{1}{2}\,\mathrm{id}^2\mathcal{F}\Big(\mathcal{F}^{-1}(\mathrm{id}^{-1}f)\,\mathcal{F}^{-1}(\mathrm{id}^{-1}g)\Big) - \frac{1}{2}\,\mathcal{F}\Big(\mathcal{F}^{-1}(\mathrm{id}f)\,\mathcal{F}^{-1}(\mathrm{id}^{-1}g)\Big) \\ &\quad - \frac{1}{2}\,\mathcal{F}\Big(\mathcal{F}^{-1}(\mathrm{id}^{-1}f)\,\mathcal{F}^{-1}(\mathrm{id}\,g)\Big) \\ &= g *_{\mathrm{VV}_1} f\,, \end{aligned} \tag{C.10b}$$

$$\begin{aligned} f *_{\mathrm{VS}} g &= \frac{1}{2}\,\mathrm{id}\,\mathcal{F}\Big(\mathcal{F}^{-1}(\mathrm{id}^{-1}f)\,\mathcal{F}^{-1}(g)\Big) + \frac{1}{2}\,\mathrm{id}^{-1}\,\mathcal{F}\Big(\mathcal{F}^{-1}(\mathrm{id}f)\,\mathcal{F}^{-1}(g)\Big) \\ &\quad - \mathrm{id}^{-1}\,\mathcal{F}\Big(\mathcal{F}^{-1}(\mathrm{id}^{-1}f)\,\mathcal{F}^{-1}(\mathrm{id}^2 g)\Big) \\ &= g *_{\mathrm{SV}} f\,, \end{aligned} \tag{C.10c}$$

$$\begin{aligned} f *_{\mathrm{TV}} g &= \frac{1}{4}\,\mathrm{id}^3\,\mathcal{F}\Big(\mathcal{F}^{-1}(\mathrm{id}^{-2}f)\,\mathcal{F}^{-1}(\mathrm{id}^{-1}g)\Big) - \frac{1}{4}\,\mathrm{id}^{-1}\,\mathcal{F}\Big(\mathcal{F}^{-1}(\mathrm{id}^2 f)\,\mathcal{F}^{-1}(\mathrm{id}^{-1}g)\Big) \\ &\quad + \frac{1}{4}\,\mathrm{id}^{-1}\,\mathcal{F}\Big(\mathcal{F}^{-1}(\mathrm{id}^{-2}f)\,\mathcal{F}^{-1}(\mathrm{id}^3 g)\Big) - \frac{1}{2}\,\mathrm{id}\,\mathcal{F}\Big(\mathcal{F}^{-1}(\mathrm{id}^{-2}f)\,\mathcal{F}^{-1}(\mathrm{id}\,g)\Big) \\ &= g *_{\mathrm{VT}} f\,, \end{aligned} \tag{C.10d}$$

$$\begin{aligned} f *_{\mathrm{VV}_2} g &= \frac{1}{4}\,\mathrm{id}^2\,\mathcal{F}\Big(\mathcal{F}^{-1}(\mathrm{id}^{-1}f)\,\mathcal{F}^{-1}(\mathrm{id}^{-1}\,g)\Big) - \frac{1}{4}\,\mathrm{id}^{-2}\,\mathcal{F}\Big(\mathcal{F}^{-1}(\mathrm{id}^3 f)\,\mathcal{F}^{-1}(\mathrm{id}^{-1}g)\Big) \\ &\quad + \frac{1}{2}\,\mathrm{id}^{-2}\,\mathcal{F}\Big(\mathcal{F}^{-1}(\mathrm{id}\,f)\,\mathcal{F}^{-1}(\mathrm{id}\,g)\Big) - \frac{1}{4}\,\mathrm{id}^{-2}\,\mathcal{F}\Big(\mathcal{F}^{-1}(\mathrm{id}^{-1}f)\,\mathcal{F}^{-1}(\mathrm{id}^3\,g)\Big) \\ &= g *_{\mathrm{VV}_2} f\,. \end{aligned} \tag{C.10e}$$

Note that, in contrast to the standard convolution, generalized convolutions do not commute in general. If generalized convolutions commute can easily be checked by looking at their representations in terms of tuples: A generalized convolution commutes if (and only if) for each tuple $(\alpha,(\ell,m,n))$ (with $n \neq m$) in its representation, there is also the tuple $(\alpha,(\ell,n,m))$. It follows that the scalar-scalar and type 2 vector-vector convolutions commute (since the scalar-scalar convolution is the ordinary convolution, it is clear that it commutes), while all others do not commute. However, some generalized convolutions are not independent: One has $f *_{\mathrm{SV}} g = g *_{\mathrm{VS}} f$ and $f *_{\mathrm{VT}} g = g *_{\mathrm{TV}} f$, since for those two generalized convolutions, for each tuple $(\alpha,(\ell,m,n))$, the tuple $(\alpha,(\ell,n,m))$ is in the representation of the respective *other* generalized convolution.

## C.2 Validity Check of the Generalized Convolution Theorems

In order to check the correctness of the generalized convolution theorems, we can compare the results to the results obtained by direct integration. We will do this for two different cases: In the first case, we will consider the generalized convolution of two Gaussians, which is finite, while in the second case, we will consider the generalized convolution of two cosines, which diverges. These choices of the functions to be convolved are convenient since in both cases the angular integrals can be solved exactly, so that only an integral over the modulus of the spatial momentum remains.

With the prefactor $\alpha_{\mathrm{A}}(|\boldsymbol{p}|, |\boldsymbol{q}|, |\boldsymbol{p}-\boldsymbol{q}|)$ defining the type of generalized convolution, we have for an arbitrary generalized convolution of type A of two functions $f_1$ and $f_2$:

$$
\begin{aligned}
g(|\boldsymbol{p}|) &= (f_1 *_{\mathrm{A}} f_2)(|\boldsymbol{p}|) \\
&= \int_{\boldsymbol{q}} \alpha_{\mathrm{A}}(|\boldsymbol{p}|, |\boldsymbol{q}|, |\boldsymbol{p}-\boldsymbol{q}|)\, f_1(|\boldsymbol{q}|) f_2(|\boldsymbol{p}-\boldsymbol{q}|) \\
&= \frac{1}{4\pi^2} \int_0^\infty \mathrm{d}q\, q^2 f_1(q) \int_{-1}^{1} \mathrm{d}x\, \alpha_{\mathrm{A}}\Big(p, q, \sqrt{p^2+q^2-2\,p\,q\,x}\Big)\, f_2\Big(\sqrt{p^2+q^2-2\,p\,q\,x}\Big) \\
&= \frac{1}{4\pi^2\, p} \int_0^\infty \mathrm{d}q\, q\, f_1(q) \underbrace{\int_{|p-q|}^{p+q} \mathrm{d}k\, k\, \alpha_{\mathrm{A}}(p, q, k)\, f_2(k)}_{:=\tilde{f}_2(p,q)} \\
&= \frac{1}{4\pi^2\, p} \int_0^\infty \mathrm{d}q\, q\, f_1(q) \tilde{f}_2(p, q) \\
&= g(p)
\end{aligned}
\tag{C.11}
$$

with

$$
\begin{aligned}
\alpha_{\mathrm{S}}(p, q, k) &= 1\,, \\
\alpha_{\mathrm{VV}_1}(p, q, k) &= \frac{p^2-q^2-k^2}{2\,q\,k}\,, \\
\alpha_{\mathrm{VS}}(p, q, k) &= \frac{p^2+q^2-k^2}{2\,p\,q}\,, \\
\alpha_{\mathrm{SV}}(p, q, k) &= \frac{p^2-q^2+k^2}{2\,p\,k}\,, \\
\alpha_{\mathrm{TV}}(p, q, k) &= \frac{(p^2-k^2)^2-q^4}{4\,p\,q^2\,k} = \alpha_{\mathrm{VS}}(p, q, k)\, \alpha_{\mathrm{VV}_1}(p, q, k)\,, \\
\alpha_{\mathrm{VT}}(p, q, k) &= \frac{(p^2-q^2)^2-k^4}{4\,p\,q\,k^2} = \alpha_{\mathrm{VV}_1}(p, q, k)\, \alpha_{\mathrm{SV}}(p, q, k)\,, \\
\alpha_{\mathrm{VV}_2}(p, q, k) &= \frac{p^4-(q^2-k^2)^2}{4\,p^2\,q\,k} = \alpha_{\mathrm{VS}}(p, q, k)\, \alpha_{\mathrm{SV}}(p, q, k)\,.
\end{aligned}
$$

It follows that

$$(f_1 *_{\mathrm{A}} f_2)(p) = \frac{1}{4\pi^2} \sum_{(\alpha,(\ell,m,n))\in\mathcal{A}} \alpha\, p^{\ell-1} \int_0^\infty \mathrm{d}q\, q^{m+1} f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, k^{n+1} f_2(k)\,, \tag{C.12}$$

or explicitly:

$$(f_1 *_{\mathrm{SS}} f_2)(p) = \frac{1}{4\pi^2\, p} \int_0^\infty \mathrm{d}q\, q\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, k\, f_2(k)\,, \tag{C.13a}$$

$$(f_1 *_{\mathrm{VV_1}} f_2)(p) = \frac{1}{8\pi^2} \bigg[ p \int_0^\infty \mathrm{d}q\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, f_2(k) - \frac{1}{p} \int_0^\infty \mathrm{d}q\, q^2\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, f_2(k) - \frac{1}{p} \int_0^\infty \mathrm{d}q\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, k^2\, f_2(k) \bigg]\,, \tag{C.13b}$$

$$(f_1 *_{\mathrm{VS}} f_2)(p) = \frac{1}{8\pi^2} \bigg[ \int_0^\infty \mathrm{d}q\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, k\, f_2(k) + \frac{1}{p^2} \int_0^\infty \mathrm{d}q\, q^2\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, k\, f_2(k) - \frac{1}{p^2} \int_0^\infty \mathrm{d}q\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, k^3\, f_2(k) \bigg]\,, \tag{C.13c}$$

$$(f_1 *_{\mathrm{SV}} f_2)(p) = \frac{1}{8\pi^2} \bigg[ \int_0^\infty \mathrm{d}q\, q\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, f_2(k) - \frac{1}{p^2} \int_0^\infty \mathrm{d}q\, q^3\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, f_2(k) + \frac{1}{p^2} \int_0^\infty \mathrm{d}q\, q\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, k^2\, f_2(k) \bigg]\,, \tag{C.13d}$$

$$(f_1 *_{\mathrm{TV}} f_2)(p) = \frac{1}{8\pi^2} \bigg[ \frac{p^2}{2} \int_0^\infty \mathrm{d}q\, \frac{f_1(q)}{q} \int_{|p-q|}^{p+q} \mathrm{d}k\, f_2(k) + \frac{1}{2p^2} \int_0^\infty \mathrm{d}q\, \frac{f_1(q)}{q} \int_{|p-q|}^{p+q} \mathrm{d}k\, k^4\, f_2(k) - \frac{1}{2p^2} \int_0^\infty \mathrm{d}q\, q^3\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, f_2(k) - \int_0^\infty \mathrm{d}q\, \frac{f_1(q)}{q} \int_{|p-q|}^{p+q} \mathrm{d}k\, k^2\, f_2(k) \bigg]\,, \tag{C.13e}$$

$$(f_1 *_{\mathrm{VT}} f_2)(p) = \frac{1}{8\pi^2} \bigg[ \frac{p^2}{2} \int_0^\infty \mathrm{d}q\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, \frac{f_2(k)}{k} + \frac{1}{2p^2} \int_0^\infty \mathrm{d}q\, q^4\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, \frac{f_2(k)}{k} - \frac{1}{2p^2} \int_0^\infty \mathrm{d}q\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, k^3\, f_2(k) - \int_0^\infty \mathrm{d}q\, q^2\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, \frac{f_2(k)}{k} \bigg]\,, \tag{C.13f}$$

$$(f_1 *_{\mathrm{VV_2}} f_2)(p) = \frac{1}{8\pi^2} \bigg[ \frac{p}{2} \int_0^\infty \mathrm{d}q\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, f_2(k) - \frac{1}{2p^3} \int_0^\infty \mathrm{d}q\, q^4\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, f_2(k) - \frac{1}{2p^3} \int_0^\infty \mathrm{d}q\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, k^4\, f_2(k) + \frac{1}{p^3} \int_0^\infty \mathrm{d}q\, q^2\, f_1(q) \int_{|p-q|}^{p+q} \mathrm{d}k\, k^2\, f_2(k) \bigg]\,. \tag{C.13g}$$

Although not all generalized convolutions can be found analytically, it turns out that the zero mode

$$
\begin{aligned}
(f_1 *_{\mathrm{A}} f_2)(0) &= \frac{1}{2\pi^2} \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} \int_0^\infty \mathrm{d}q\, q\, f_1(q) \lim_{p\to 0} \left\{ \left( p\, \frac{\partial}{\partial k} \right)^{2n} \Big[ k\, \alpha_{\mathrm{A}}(p,q,k)\, f_2(k) \Big] \bigg|_{k=q} \right\} \\
&= \frac{1}{2\pi^2} \int_0^\infty \mathrm{d}q\, q^2\, \alpha_{\mathrm{A}}(0,q,q)\, f_1(q)\, f_2(q) + \ldots
\end{aligned} \tag{C.14}
$$

can indeed be found analytically for each type of generalized convolution defined above.[4] In fact, one has $\alpha_{\mathrm{SS}}(0,q,q) = -\alpha_{\mathrm{VV}_1}(0,q,q) = -1$ and $\alpha_{\mathrm{VS}}(0,q,q) = \alpha_{\mathrm{SV}}(0,q,q) = \alpha_{\mathrm{TV}}(0,q,q) = \alpha_{\mathrm{VT}}(0,q,q) = \alpha_{\mathrm{VV}_2}(0,q,q) = 0$.

Note that for a finite spatial momentum cutoff $\Lambda$, we have

$$
(f_1 *_{\mathrm{A}} f_2)(p) = \frac{1}{4\pi^2 p} \int_0^\Lambda \mathrm{d}q\, q\, f_1(q) \int_{|p-q|}^{\min(\Lambda, p+q)} \mathrm{d}k\, k\, \alpha_{\mathrm{A}}(p,q,k)\, f_2(k)\,. \tag{C.15}
$$

### Generalized Convolutions of Two Gaussians

We define the (unnormalized) Gaussians

$$
f_i(|\boldsymbol{p}|) = \mathrm{e}^{-\lambda_i^2 |\boldsymbol{p}|^2} \tag{C.16}
$$

($i = 1,2$) with vanishing mean and variance $1/(2\lambda_i^2)$.

- **Scalar-Scalar Convolution** For the scalar-scalar (i. e. standard) convolution, we obtain:
$$
\tilde{f}_2(p,q) = \int_{|p-q|}^{p+q} \mathrm{d}k\, k\, f_2(k) = \frac{1}{2\lambda_2^2} \left[ \mathrm{e}^{-\lambda_2^2 (p-q)^2} - \mathrm{e}^{-\lambda_2^2 (p+q)^2} \right] .
$$
In fact, in this case we can even solve the remaining integral analytically to obtain
$$
g(p) = \frac{1}{\left[ 4\pi (\lambda_1^2 + \lambda_2^2)^2 \right]^{3/2}}\ \mathrm{e}^{-\frac{\lambda_1^2 \lambda_2^2}{\lambda_1^2 + \lambda_2^2} p^2} .
$$
- **Type 1 Vector-Vector Convolution** For the type 1 vector-vector convolution, we obtain:
$$
\begin{aligned}
\tilde{f}_2(p,q) &= \frac{1}{2\,q} \int_{|p-q|}^{p+q} \mathrm{d}k\, (p^2 - q^2 - k^2)\, f_2(k) \\
&= \frac{1}{8\lambda_2^3\, q} \Bigg\{ \sqrt{\pi} \Big[ 2\lambda_2^2 (p^2 - q^2) - 1 \Big] \Big[ \mathrm{erf}(\lambda_2 (p+q)) - \mathrm{erf}(\lambda_2 |p-q|) \Big] \\
&\qquad + 2\, \lambda_2 \left[ (p+q)\, \mathrm{e}^{-\lambda_2^2 (p+q)^2} - |p-q|\, \mathrm{e}^{-\lambda_2^2 (p-q)^2} \right] \Bigg\}
\end{aligned}
$$
with the error function
$$
\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \mathrm{d}t\ \mathrm{e}^{-t^2} .
$$

[4] Only for the type-2 vector-vector convolution the second term in the expansion is needed since $p$ appears as a square in the denominator of $\alpha_{\mathrm{VV}_2}(p,q,k)$.

- **Vector-Scalar Convolution** For the vector-scalar convolution, we obtain:

$$\begin{aligned}\tilde{f}_2(p,q) &= \frac{1}{2\,p\,q}\int_{|p-q|}^{p+q} \mathrm{d}k\, k\,(p^2+q^2-k^2)\, f_2(k)\\ &= \frac{1}{4\,\lambda_2^4\,p\,q}\left[(2\,\lambda_2^2\,p\,q+1)\,\mathrm{e}^{-\lambda_2^2\,(p+q)^2} + (2\,\lambda_2^2\,p\,q-1)\,\mathrm{e}^{-\lambda_2^2\,(p-q)^2}\right].\end{aligned}$$

In fact, in this case we can even solve the remaining integral analytically to obtain

$$\begin{aligned}g(p) = \frac{1}{16\pi^2\lambda_2^4(\lambda_1^2+\lambda_2^2)^{3/2}p^2}\Bigg\{&\sqrt{\pi}\Big[2\lambda_2^4p^2-(\lambda_1^2+\lambda_2^2)\Big]\mathrm{erf}\Bigg(\frac{\lambda_2^2}{\sqrt{\lambda_1^2+\lambda_2^2}}\,p\Bigg)\,\mathrm{e}^{-\frac{\lambda_1^2\lambda_2^2}{\lambda_1^2+\lambda_2^2}\,p^2}\\ &+2\lambda_2^2\sqrt{\lambda_1^2+\lambda_2^2}\,p\,\mathrm{e}^{-\lambda_2^2\,p^2}\Bigg\}.\end{aligned}$$

- **Scalar-Vector Convolution** For the scalar-vector convolution, we obtain:

$$\begin{aligned}\tilde{f}_2(p,q) &= \frac{1}{2\,p}\int_{|p-q|}^{p+q} \mathrm{d}k\,(p^2-q^2+k^2)\, f_2(k)\\ &= \frac{1}{8\lambda_2^3\,p}\Bigg\{\sqrt{\pi}\Big[2\lambda_2^2(p^2-q^2)+1\Big]\Big[\mathrm{erf}(\lambda_2(p+q))-\mathrm{erf}(\lambda_2|p-q|)\Big]\\ &\qquad -2\,\lambda_2\left[(p+q)\,\mathrm{e}^{-\lambda_2^2\,(p+q)^2} - |p-q|\,\mathrm{e}^{-\lambda_2^2(p-q)^2}\right]\Bigg\}.\end{aligned}$$

- **Tensor-Vector Convolution** For the tensor-vector convolution, we obtain:

$$\begin{aligned}\tilde{f}_2(p,q) &= \frac{1}{2\,p\,q^2}\int_{|p-q|}^{p+q} \mathrm{d}k\,\Big[(p^2-k^2)^2-q^4\Big]\, f_2(k)\\ &= \frac{1}{32\lambda_2^5\,p\,q^2}\Bigg\{\sqrt{\pi}\Big[4\lambda_2^4(p^4-q^4)-4\lambda_2^2\,p^2+3\Big]\Big[\mathrm{erf}(\lambda_2(p+q))-\mathrm{erf}(\lambda_2|p-q|)\Big]\\ &\qquad +2\lambda_2\Big[\;(p+q)\Big[2\lambda_2^2\,(p^2-2\,p\,q-q^2)-3\Big]\,\mathrm{e}^{-\lambda_2^2(p+q)^2}\\ &\qquad\qquad -|p-q|\Big[2\lambda_2^2\,(p^2+2\,p\,q-q^2)-3\Big]\,\mathrm{e}^{-\lambda_2^2(p-q)^2}\Big]\Bigg\}.\end{aligned}$$

- **Vector-Tensor Convolution** For the vector-tensor convolution, we obtain:

$$\begin{aligned}\tilde{f}_2(p,q) &= \frac{1}{4p\,q^2}\int_{|p-q|}^{p+q} \mathrm{d}k\,\frac{(p^2-q^2)^2-k^4}{k}\, f_2(k)\\ &= \frac{1}{8\lambda_2^4\,p\,q}\Bigg\{\;\lambda_2^4(p^2-q^2)^2\Big[\mathrm{Ei}\Big(-\lambda_2^2(p+q)^2\Big)-\mathrm{Ei}\Big(-\lambda_2^2(p-q)^2\Big)\Big]\\ &\qquad +\Big[\lambda_2^2(p+q)^2+1\Big]\,\mathrm{e}^{-\lambda_2^2(p+q)^2} - \Big[\lambda_2^2(p-q)^2+1\Big]\,\mathrm{e}^{-\lambda_2^2(p-q)^2}\Bigg\}\end{aligned}$$

with the exponential integral

$$\mathrm{Ei}(x) = \int_{-\infty}^{x} \mathrm{d}t \, \frac{e^t}{t} \, .$$

- **Type 2 Vector-Vector Convolution** For the type 2 vector-vector convolution, we obtain:

$$\begin{aligned}
\tilde{f}_2(p,q) &= \frac{1}{4p^2 q} \int_{|p-q|}^{p+q} \mathrm{d}k \left[p^4 - (q^2 - k^2)^2\right] f_2(k) \\
&= \frac{1}{32\lambda_2^5 \, p^2 \, q} \Bigg\{ \sqrt{\pi} \left[4\lambda_2^4 (p^4 - q^4) + 4\lambda_2^2 \, q^2 - 3\right] \left[\mathrm{erf}(\lambda_2 (p+q)) - \mathrm{erf}(\lambda_2 |p-q|)\right] \\
&\qquad + 2\lambda_2 \Big[ \ (p+q) \left[2\lambda_2^2 \, (p^2 + 2\,p\,q - q^2) + 3\right] \mathrm{e}^{-\lambda_2^2 (p+q)^2} \\
&\qquad\qquad - |p-q| \left[2\lambda_2^2 \, (p^2 - 2\,p\,q - q^2) + 3\right] \mathrm{e}^{-\lambda_2^2 (p-q)^2} \Big] \Bigg\} \, .
\end{aligned}$$

We have

$$g(0) = -\frac{1}{3[4\pi(\lambda_1^2 + \lambda_2^2)^2]^{3/2}} \, .$$

Table (C.1) shows a comparison of generalized convolutions of two Gaussians (C.16) calculated by employing the convolution theorem and by direct integration.

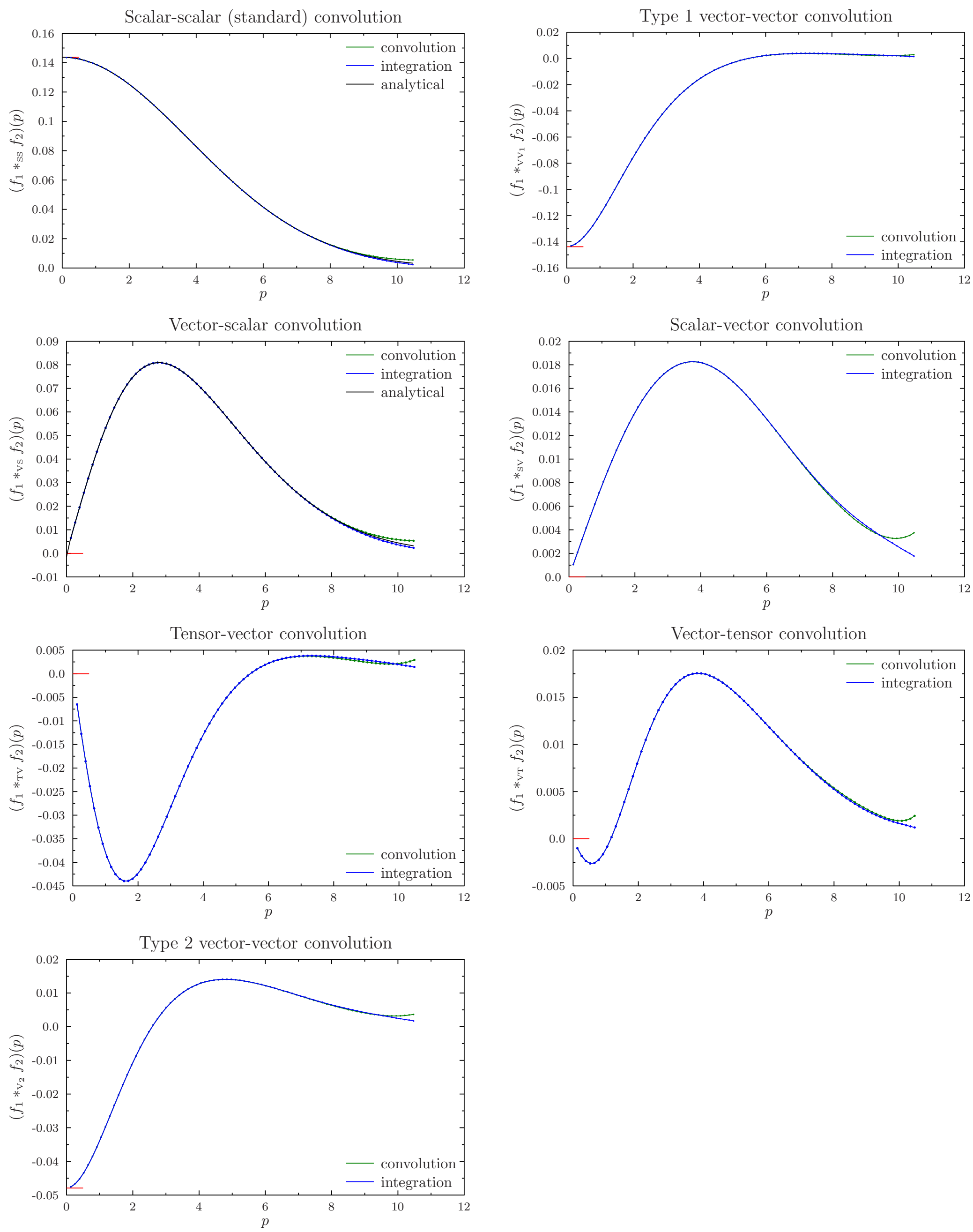



Figure C.1: Comparison of generalized convolutions of two Gaussians (C.16) calculated by employing the convolution theorem and by direct integration for $a_{\mathrm{S}} = 0.3$, $N_{\mathrm{S}} = 80$, $\lambda_1 = 0.2$, $\lambda_2 = 0.5$ (for the standard and for the vector-scalar convolution, the result can be obtained analytically and is for these cases plotted as well). The curves lie on top of each other almost exactly except for large momenta, where small deviations occur. The deviations can be decreased by decreasing the width of the Gaussians. The small red tic indicates the value at $p = 0$.

# Appendix D

# Properties of the Two-Point Functions

In this appendix, we state properties of the photon and fermion two-point functions used throughout this work which follow from various symmetries.

## D.1 Photons

### D.1.1 Reality

Since gauge fields are elements of the Lie algebra of the respective gauge group, the expectation values of the photon field are real (since the Lie algebra of U(1) are the real numbers), i. e. the photon field operator is hermitean:

$$A_\mu(x) = A_\mu(x)^\dagger \,.$$

#### Spectral Function

It hence follows:

$$\begin{aligned}
\rho^{(\mathrm{g})}_{\mu\nu}(x,y)^* &= \Big\{ \mathrm{i}\, \langle [A_\mu(x), A_\nu(y)] \rangle \Big\}^* \\
&= -\,\mathrm{i}\, \langle [A_\mu(x), A_\nu(y)]^\dagger \rangle \\
&= -\,\mathrm{i}\, \langle [A_\nu(y)^\dagger, A_\mu(x)^\dagger] \rangle \\
&= -\,\mathrm{i}\, \langle [A_\nu(y), A_\mu(x)] \rangle \\
&= \mathrm{i}\, \langle [A_\mu(x), A_\nu(y)] \rangle \\
&= \rho^{(\mathrm{g})}_{\mu\nu}(x,y) \,,
\end{aligned}$$

i. e. the photon spectral function in real space is real.

Expressed by its Fourier transform, one has:

$$\rho^{(\mathrm{g})}_{\mu\nu}(x,y)^* = \int_{\boldsymbol{p}} \rho^{(\mathrm{g})}_{\mu\nu}(x^0,y^0;\boldsymbol{p})^*\, \mathrm{e}^{-\,\mathrm{i}\,\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} = \int_{\boldsymbol{p}} \rho^{(\mathrm{g})}_{\mu\nu}(x^0,y^0;\boldsymbol{p})\, \mathrm{e}^{\mathrm{i}\,\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} = \rho^{(\mathrm{g})}_{\mu\nu}(x,y) \,,$$

so

$$\rho^{(\mathrm{g})}_{\mu\nu}(x^0, y^0; \boldsymbol{p}) = \rho^{(\mathrm{g})}_{\mu\nu}(x^0, y^0; -\boldsymbol{p})^* \,.$$

For the isotropic components, it follows that

$$\begin{aligned}
\rho^{(\mathrm{g})}_{\mathrm{S}}(x^0, y^0; p) &= \rho^{(\mathrm{g})}_{\mathrm{S}}(x^0, y^0; p)^* \,, \\
\rho^{(\mathrm{g})}_{\mathrm{V}_1}(x^0, y^0; p) &= -\rho^{(\mathrm{g})}_{\mathrm{V}_1}(x^0, y^0; p)^* \,, \\
\rho^{(\mathrm{g})}_{\mathrm{V}_2}(x^0, y^0; p) &= -\rho^{(\mathrm{g})}_{\mathrm{V}_2}(x^0, y^0; p)^* \,, \\
\rho^{(\mathrm{g})}_{\mathrm{T}}(x^0, y^0; p) &= \rho^{(\mathrm{g})}_{\mathrm{T}}(x^0, y^0; p)^* \,, \\
\rho^{(\mathrm{g})}_{\mathrm{L}}(x^0, y^0; p) &= \rho^{(\mathrm{g})}_{\mathrm{L}}(x^0, y^0; p)^* \,,
\end{aligned}$$

i. e. the vector components are purely imaginary, while the other components are real. It is therefore convenient to define $\rho^{(\mathrm{g})}_{\mathrm{V}_a}(x^0, y^0; p) = \mathrm{i}\, \widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_a}(x^0, y^0; p)$ $(a = 1, 2)$, so that

$$\begin{aligned}
\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(x^0, y^0; p) &= \widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(x^0, y^0; p)^* \,, \\
\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(x^0, y^0; p) &= \widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(x^0, y^0; p)^* \,,
\end{aligned}$$

i. e. these vector components are real.

### Statistical Function

Similarly, it follows that:

$$\begin{aligned}
F^{(\mathrm{g})}_{\mu\nu}(x, y)^* &= \left[\frac{1}{2} \left\langle \{A_\mu(x), A_\nu(y)\} \right\rangle\right]^* \\
&= \frac{1}{2} \left\langle \{A_\mu(x), A_\nu(y)\}^\dagger \right\rangle \\
&= \frac{1}{2} \left\langle \{A_\nu(y)^\dagger, A_\mu(x)^\dagger\} \right\rangle \\
&= \frac{1}{2} \left\langle \{A_\nu(y), A_\mu(x)\} \right\rangle \\
&= \frac{1}{2} \left\langle \{A_\mu(x), A_\nu(y)\} \right\rangle \\
&= F^{(\mathrm{g})}_{\mu\nu}(x, y) \,,
\end{aligned}$$

i. e. the photon statistical function in real space is real.

Expressed by its Fourier transform, one has:

$$F^{(\mathrm{g})}_{\mu\nu}(x, y)^* = \int_{\boldsymbol{p}} F^{(\mathrm{g})}_{\mu\nu}(x^0, y^0; \boldsymbol{p})^* \, \mathrm{e}^{-\mathrm{i}\, \boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} = \int_{\boldsymbol{p}} F^{(\mathrm{g})}_{\mu\nu}(x^0, y^0; \boldsymbol{p}) \, \mathrm{e}^{\mathrm{i}\, \boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} = F^{(\mathrm{g})}_{\mu\nu}(x, y) \,,$$

so

$$F^{(\mathrm{g})}_{\mu\nu}(x^0, y^0; \boldsymbol{p}) = F^{(\mathrm{g})}_{\mu\nu}(x^0, y^0; -\boldsymbol{p})^* \,.$$

For the isotropic components, it follows that

$$
\begin{aligned}
F^{(\mathrm{g})}_{\mathrm{S}}(x^0,y^0;p) &= F^{(\mathrm{g})}_{\mathrm{S}}(x^0,y^0;p)^* \,, \\
F^{(\mathrm{g})}_{\mathrm{V}_1}(x^0,y^0;p) &= -F^{(\mathrm{g})}_{\mathrm{V}_1}(x^0,y^0;p)^* \,, \\
F^{(\mathrm{g})}_{\mathrm{V}_2}(x^0,y^0;p) &= -F^{(\mathrm{g})}_{\mathrm{V}_2}(x^0,y^0;p)^* \,, \\
F^{(\mathrm{g})}_{\mathrm{T}}(x^0,y^0;p) &= F^{(\mathrm{g})}_{\mathrm{T}}(x^0,y^0;p)^* \,, \\
F^{(\mathrm{g})}_{\mathrm{L}}(x^0,y^0;p) &= F^{(\mathrm{g})}_{\mathrm{L}}(x^0,y^0;p)^* \,,
\end{aligned}
$$

i. e. the vector components are purely imaginary, while the other components are real. It is therefore convenient to define $F^{(\mathrm{g})}_{\mathrm{V}_a}(x^0,y^0;p) = \mathrm{i}\,\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_a}(x^0,y^0;p)$ $(a=1,2)$, so that

$$
\begin{aligned}
\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(x^0,y^0;p) &= \widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(x^0,y^0;p)^* \,, \\
\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(x^0,y^0;p) &= \widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(x^0,y^0;p)^* \,,
\end{aligned}
$$

i. e. these vector components are real.

#### Feynman Propagator

For the Feynman propagator, one then obtains:

$$
\begin{aligned}
D_{\mu\nu}(x,y)^* &= \left[F^{(\mathrm{g})}_{\mu\nu}(x,y) - \frac{\mathrm{i}}{2}\,\mathrm{sgn}(x^0-y^0)\rho^{(\mathrm{g})}_{\mu\nu}(x,y)\right]^* \\
&= F^{(\mathrm{g})}_{\mu\nu}(x,y)^* + \frac{\mathrm{i}}{2}\,\mathrm{sgn}(x^0-y^0)\rho^{(\mathrm{g})}_{\mu\nu}(x,y)^* \\
&= F^{(\mathrm{g})}_{\mu\nu}(x,y) + \frac{\mathrm{i}}{2}\,\mathrm{sgn}(x^0-y^0)\rho^{(\mathrm{g})}_{\mu\nu}(x,y) \,.
\end{aligned}
$$

In fact, the statistical function is the real part of the propagator, while the spectral function is (up to a factor) its imaginary part.

### D.1.2 Exchange of Arguments

#### Spectral Function

One has[1]

$$
\rho^{(\mathrm{g})}_{\mu\nu}(x,y) = \mathrm{i}\langle[A_\mu(x),A_\nu(y)]\rangle = -\,\mathrm{i}\langle[A_\nu(y),A_\mu(x)]\rangle = -\rho^{(\mathrm{g})}_{\nu\mu}(y,x) \,,
$$

so

$$
\rho^{(\mathrm{g})}_{\mu\nu}(x^0,y^0;\boldsymbol{p}) = -\rho^{(\mathrm{g})}_{\nu\mu}(y^0,x^0;-\boldsymbol{p}) \,.
$$

[1]This can also be written in a coordinate-independent fashion as $\rho^{(\mathrm{g})}(x,y) = -\rho^{(\mathrm{g})}(y,x)^\top$ to make it more closely resemble the case of the fermions.

For the isotropic components, this translates to:

$$
\begin{aligned}
\rho^{(\mathrm{g})}_{\mathrm{S}}(x^0, y^0; p) &= -\rho^{(\mathrm{g})}_{\mathrm{S}}(y^0, x^0; p)\,, \\
\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(x^0, y^0; p) &= \widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(y^0, x^0; p)\,, \\
\rho^{(\mathrm{g})}_{\mathrm{T}}(x^0, y^0; p) &= -\rho^{(\mathrm{g})}_{\mathrm{T}}(y^0, x^0; p)\,, \\
\rho^{(\mathrm{g})}_{\mathrm{L}}(x^0, y^0; p) &= -\rho^{(\mathrm{g})}_{\mathrm{L}}(y^0, x^0; p)\,.
\end{aligned}
$$

Note that this implies that at equal times, the scalar, transverse and longitudinal components have to vanish (in accordance with their initial conditions), while this is not the case for the vector components. Instead, one has $\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(x^0, x^0; p) = \widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(x^0, x^0; p)$, i. e., at equal times, the two vector components are equal. Note, however, that according to the equal-time commutation relations (3.4.2), it follows that the equal-time vector components vanish identically, like the other components.

### Statistical Function

One has[2]

$$
F^{(\mathrm{g})}_{\mu\nu}(x, y) = \langle \{A_\mu(x), A_\nu(y)\}\rangle = \langle \{A_\nu(y), A_\mu(x)\}\rangle = F^{(\mathrm{g})}_{\nu\mu}(y, x)\,,
$$

so

$$
F^{(\mathrm{g})}_{\mu\nu}(x^0, y^0; \boldsymbol{p}) = F^{(\mathrm{g})}_{\nu\mu}(y^0, x^0; -\boldsymbol{p})\,.
$$

For the isotropic components, this translates to:

$$
\begin{aligned}
F^{(\mathrm{g})}_{\mathrm{S}}(x^0, y^0; p) &= F^{(\mathrm{g})}_{\mathrm{S}}(y^0, x^0; p)\,, \\
\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_1}(x^0, y^0; p) &= -\widetilde{F}^{(\mathrm{g})}_{\mathrm{V}_2}(y^0, x^0; p)\,, \\
F^{(\mathrm{g})}_{\mathrm{T}}(x^0, y^0; p) &= F^{(\mathrm{g})}_{\mathrm{T}}(y^0, x^0; p)\,, \\
F^{(\mathrm{g})}_{\mathrm{L}}(x^0, y^0; p) &= F^{(\mathrm{g})}_{\mathrm{L}}(y^0, x^0; p)\,.
\end{aligned}
$$

### Feynman Propagator

One has

$$
\begin{aligned}
D_{\mu\nu}(x, y) &= F^{(\mathrm{g})}_{\mu\nu}(x, y) - \frac{\mathrm{i}}{2}\,\mathrm{sgn}(x^0 - y^0)\rho^{(\mathrm{g})}_{\mu\nu}(x, y) \\
&= F^{(\mathrm{g})}_{\nu\mu}(y, x) + \frac{\mathrm{i}}{2}\,\mathrm{sgn}(x^0 - y^0)\rho^{(\mathrm{g})}_{\nu\mu}(y, x) \\
&= F^{(\mathrm{g})}_{\nu\mu}(y, x) - \frac{\mathrm{i}}{2}\,\mathrm{sgn}(y^0 - x^0)\rho^{(\mathrm{g})}_{\nu\mu}(y, x) \\
&= D_{\nu\mu}(y, x)\,,
\end{aligned}
$$

[2]This can also be written in a coordinate-independent fashion as $F^{(\mathrm{g})}(x, y) = F^{(\mathrm{g})}(y, x)^\top$ to make it more closely resemble the case of the fermions.

or equivalently from the definition in terms of field operators:

$$\begin{aligned} D_{\mu\nu}(x,y) &= \Theta(x^0-y^0)\langle A_\mu(x)A_\nu(y)\rangle + \Theta(y^0-x^0)\langle A_\nu(y)A_\mu(x)\rangle \\ &= \Theta(y^0-x^0)\langle A_\nu(y)A_\mu(x)\rangle + \Theta(x^0-y^0)\langle A_\mu(x)A_\mu(y)\rangle \\ &= D_{\nu\mu}(y,x)\,. \end{aligned}$$

Further, if one has time- as well as space-translation invariance like in vacuum or thermal equilibrium,

$$D_{\mu\nu}(x-y) = D_{\nu\mu}(y-x) \overset{\text{CPT}}{\longmapsto} D_{\nu\mu}(x-y)\,.$$

From CPT invariance[3], it hence follows that the propagator is symmetric in its Lorentz indices. Note that the assumption of time- and space-translation invariance is crucial, since otherwise the two arguments of the propagator are not related to each other (like out-of-equilibrium). One would then have

$$D_{\mu\nu}(x,y) = D_{\nu\mu}(y,x) \overset{\text{CPT}}{\longmapsto} D_{\nu\mu}(-y,-x)\,,$$

from which one cannot derive a statement about the symmetry with respect to the Lorentz indices. It hence follows that out-of-equilibrium, the photon propagator is not constrained to be symmetric in its Lorentz indices, and in particular it follows that there are *two* independent vector components in general.

### D.1.3 Parity Transformation

Consider a timelike vector $n$ (i.e. $n^2 > 0$). Then the parity transform of a vector $v$ is given by

$$v \mapsto Pv = P\Big\{(n\cdot v)\,n + \Big[v-(n\cdot v)\,n\Big]\Big\} = (n\cdot v)\,n - \Big[v-(n\cdot v)\,n\Big]$$

where $P$ denotes the (fundamental) representation of the parity transformation on vectors.

Let us denote the parity transform of an arbitrary vector (or covector) $v$ as $\overline{v}$, i.e. $\overline{v}_0 = v_0$ and $\overline{v}_i = -\overline{v}_i$. Then under a parity transformation, the photon field transforms as

$$A_\mu(x) \mapsto \mathcal{P}A_\mu(x)\,\mathcal{P}^\dagger = \overline{A}_\mu(\overline{x})\,.$$

#### Spectral Function

The photon spectral function then transforms as

$$\rho^{(\mathrm{g})}_{\mu\nu}(x,y) \mapsto \overline{\rho^{(\mathrm{g})}_{\mu\nu}}(\overline{x},\overline{y})\,,$$

[3]Actually, since the propagator transforms trivially under charge conjugation, only parity and time reflection are necessary for our purposes, but CPT is *always* a symmetry of our theory.

or componentwise:

$$\begin{aligned}
\rho^{(\mathrm{g})}_{00}(x^0,y^0;\boldsymbol{x}-\boldsymbol{y}) &\mapsto \rho^{(\mathrm{g})}_{00}(x^0,y^0;-(\boldsymbol{x}-\boldsymbol{y}))\,,\\
\rho^{(\mathrm{g})}_{i0}(x^0,y^0;\boldsymbol{x}-\boldsymbol{y}) &\mapsto -\rho^{(\mathrm{g})}_{i0}(x^0,y^0;-(\boldsymbol{x}-\boldsymbol{y}))\,,\\
\rho^{(\mathrm{g})}_{0i}(x^0,y^0;\boldsymbol{x}-\boldsymbol{y}) &\mapsto -\rho^{(\mathrm{g})}_{0i}(x^0,y^0;-(\boldsymbol{x}-\boldsymbol{y}))\,,\\
\rho^{(\mathrm{g})}_{ij}(x^0,y^0;\boldsymbol{x}-\boldsymbol{y}) &\mapsto \rho^{(\mathrm{g})}_{ij}(x^0,y^0;-(\boldsymbol{x}-\boldsymbol{y}))\,.
\end{aligned}$$

For the isotropic components, this translates to:

$$\begin{aligned}
\rho^{(\mathrm{g})}_{\mathrm{S}}(x^0,y^0;p) &\mapsto \rho^{(\mathrm{g})}_{\mathrm{S}}(x^0,y^0;p)\,,\\
\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(x^0,y^0;p) &\mapsto \widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_1}(x^0,y^0;p)\,,\\
\widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(x^0,y^0;p) &\mapsto \widetilde{\rho}^{(\mathrm{g})}_{\mathrm{V}_2}(x^0,y^0;p)\,,\\
\rho^{(\mathrm{g})}_{\mathrm{T}}(x^0,y^0;p) &\mapsto \rho^{(\mathrm{g})}_{\mathrm{T}}(x^0,y^0;p)\,,\\
\rho^{(\mathrm{g})}_{\mathrm{L}}(x^0,y^0;p) &\mapsto \rho^{(\mathrm{g})}_{\mathrm{L}}(x^0,y^0;p)\,,
\end{aligned}$$

i. e. under the assumption of spatial homogeneity, the parity transformation acts as the identity transformation on the isotropic components (but *not* on the spectral function itself).

**Statistical Function**

Obviously, the statistical function transforms in the same way as the spectral function under parity.

### D.1.4 Charge Conjugation Transformation

Under a charge conjugation transformation, the photon field transforms as

$$A_\mu(x) \mapsto \mathcal{C} A_\mu(x)\, \mathcal{C}^\dagger = -A_\mu(x)\,.$$

**Spectral Function**

It immediately follows that the photon spectral function transforms as

$$\rho^{(\mathrm{g})}_{\mu\nu}(x,y) \mapsto \rho^{(\mathrm{g})}_{\mu\nu}(x,y)\,,$$

i. e. the charge conjugation transformation acts as the identity transformation on the spectral function.

**Statistical Function**

Obviously, the statistical function transforms in the same way as the spectral function, i. e. trivially, under charge conjugation.

## D.2 Fermions

### D.2.1 Hermiticity

In the following, the identities

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0\,, \qquad (\sigma^{\mu\nu})^\dagger = \gamma^0\sigma^{\mu\nu}\gamma^0$$

for gamma matrices are useful where $\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]$.

#### Spectral Function

One has

$$\rho^{(\mathrm{f})}(x,y) = -\gamma^0\rho^{(\mathrm{f})}(y,x)^\dagger\gamma^0\,,$$

and correspondingly for the Lorentz components:

$$\begin{aligned}
\rho_{\mathrm{S}}^{(\mathrm{f})}(x,y) &= -\rho_{\mathrm{S}}^{(\mathrm{f})}(y,x)^*\,,\\
\rho_{\mathrm{V}}^{(\mathrm{f})\mu}(x,y) &= -\rho_{\mathrm{V}}^{(\mathrm{f})\mu}(y,x)^*\,,\\
\rho_{\mathrm{T}}^{(\mathrm{f})\mu\nu}(x,y) &= -\rho_{\mathrm{T}}^{(\mathrm{f})\mu\nu}(y,x)^*\,.
\end{aligned}$$

#### Statistical Function

One has

$$F^{(\mathrm{f})}(x,y) = \gamma^0F^{(\mathrm{f})}(y,x)^\dagger\gamma^0\,,$$

and correspondingly for the Lorentz components:

$$\begin{aligned}
F_{\mathrm{S}}^{(\mathrm{f})}(x,y) &= F_{\mathrm{S}}^{(\mathrm{f})}(y,x)^*\,,\\
F_{\mathrm{V}\mu}^{(\mathrm{f})}(x,y) &= F_{\mathrm{V}\mu}^{(\mathrm{f})}(y,x)^*\,,\\
F_{\mathrm{T}\mu\nu}^{(\mathrm{f})}(x,y) &= F_{\mathrm{T}\mu\nu}^{(\mathrm{f})}(y,x)^*\,.
\end{aligned}$$

#### Feynman Propagator

For the Feynman propagator, one has:

$$\begin{aligned}
\gamma^0 S(x,y)^\dagger\gamma^0 &= \gamma^0\left[F^{(\mathrm{f})}(x,y) - \frac{\mathrm{i}}{2}\,\mathrm{sgn}(x^0-y^0)\rho^{(\mathrm{f})}(x,y)\right]^\dagger\gamma^0\\
&= \gamma^0F^{(\mathrm{f})}(x,y)^\dagger\gamma^0 + \frac{\mathrm{i}}{2}\,\mathrm{sgn}(x^0-y^0)\gamma^0\rho^{(\mathrm{f})}(x,y)^\dagger\gamma^0\\
&= F^{(\mathrm{f})}(y,x) - \frac{\mathrm{i}}{2}\,\mathrm{sgn}(x^0-y^0)\rho^{(\mathrm{f})}(y,x)\\
&= F^{(\mathrm{f})}(y,x) + \frac{\mathrm{i}}{2}\,\mathrm{sgn}(y^0-x^0)\rho^{(\mathrm{f})}(y,x)\,.
\end{aligned}$$

### D.2.2 Charge Conjugation

#### Spectral Function

One has

$$
\begin{aligned}
\rho_{\mathrm{S}}^{(\mathrm{f})}(x,y) &\mapsto -\rho_{\mathrm{S}}^{(\mathrm{f})}(y,x)\,,\\
\rho_{\mathrm{V}}^{(\mathrm{f})\mu}(x,y) &\mapsto \rho_{\mathrm{V}}^{(\mathrm{f})\mu}(y,x)\,,\\
\rho_{\mathrm{T}}^{(\mathrm{f})\mu\nu}(x,y) &\mapsto \rho_{\mathrm{T}}^{(\mathrm{f})\mu\nu}(y,x)\,,
\end{aligned}
$$

so that due to charge conjugation invariance:

$$
\begin{aligned}
\rho_{\mathrm{S}}^{(\mathrm{f})}(x^0,y^0;p) &= -\rho_{\mathrm{S}}^{(\mathrm{f})}(y^0,x^0;p)\,,\\
\widetilde{\rho}_{\mathrm{V}}^{(\mathrm{f})0}(x^0,y^0;p) &= \widetilde{\rho}_{\mathrm{V}}^{(\mathrm{f})0}(y^0,x^0;p)\,,\\
\rho_{\mathrm{V}}^{(\mathrm{f})}(x^0,y^0;p) &= -\rho_{\mathrm{V}}^{(\mathrm{f})}(y^0,x^0;p)\,,\\
\rho_{\mathrm{T}}^{(\mathrm{f})}(x^0,y^0;p) &= -\rho_{\mathrm{T}}^{(\mathrm{f})}(y^0,x^0;p)\,.
\end{aligned}
$$

#### Statistical Function

One has

$$
\begin{aligned}
F_{\mathrm{S}}^{(\mathrm{f})}(x,y) &\mapsto F_{\mathrm{S}}^{(\mathrm{f})}(y,x)\,,\\
F_{\mathrm{V}}^{(\mathrm{f})\mu}(x,y) &\mapsto -F_{\mathrm{V}}^{(\mathrm{f})\mu}(y,x)\,,\\
F_{\mathrm{T}}^{(\mathrm{f})\mu\nu}(x,y) &\mapsto -F_{\mathrm{T}}^{(\mathrm{f})\mu\nu}(y,x)\,,
\end{aligned}
$$

so that due to charge conjugation invariance:

$$
\begin{aligned}
F_{\mathrm{S}}^{(\mathrm{f})}(x^0,y^0;p) &= F_{\mathrm{S}}^{(\mathrm{f})}(y^0,x^0;p)\,,\\
\widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(x^0,y^0;p) &= -\widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(y^0,x^0;p)\,,\\
F_{\mathrm{V}}^{(\mathrm{f})}(x^0,y^0;p) &= F_{\mathrm{V}}^{(\mathrm{f})}(y^0,x^0;p)\,,\\
F_{\mathrm{T}}^{(\mathrm{f})}(x^0,y^0;p) &= F_{\mathrm{T}}^{(\mathrm{f})}(y^0,x^0;p)\,.
\end{aligned}
$$

Note that it follows that $F_{\mathrm{V}}^{(\mathrm{f})\mu}(x,x) = F_{\mathrm{T}}^{(\mathrm{f})\mu\nu}(x,x) = 0$.

#### Feynman Propagator

One has

$$
\begin{aligned}
S_{\mathrm{S}}(x,y) &\mapsto S_{\mathrm{S}}(y,x)\,,\\
S_{\mathrm{V}}^{\mu}(x,y) &\mapsto -S_{\mathrm{V}}^{\mu}(y,x)\,,\\
S_{\mathrm{T}}^{\mu\nu}(x,y) &\mapsto -S_{\mathrm{T}}^{\mu\nu}(y,x)\,,
\end{aligned}
$$

so that due to charge conjugation invariance:

$$\begin{aligned}
S_{\mathrm{S}}(x^0, y^0; p) &= S_{\mathrm{S}}(y^0, x^0; p)\,,\\
S_{\mathrm{V}}^0(x^0, y^0; p) &= -S_{\mathrm{V}}^0(y^0, x^0; p)\,,\\
S_{\mathrm{V}}(x^0, y^0; p) &= S_{\mathrm{V}}(y^0, x^0; p)\,,\\
S_{\mathrm{T}}(x^0, y^0; p) &= S_{\mathrm{T}}(y^0, x^0; p)\,.
\end{aligned}$$

## D.2.3 Reality

From the hermiticity properties and the behavior under charge conjugation one can derive the behavior under complex conjugation of the Lorentz components.

### Spectral Function

One has:

$$\begin{aligned}
\rho_{\mathrm{S}}^{(\mathrm{f})}(x, y) &= \rho_{\mathrm{S}}^{(\mathrm{f})}(x, y)^*\,,\\
\rho_{\mathrm{V}}^{(\mathrm{f})\mu}(x, y) &= -\rho_{\mathrm{V}}^{(\mathrm{f})\mu}(x, y)^*\,,\\
\rho_{\mathrm{T}}^{(\mathrm{f})\mu\nu}(x, y) &= -\rho_{\mathrm{T}}^{(\mathrm{f})\mu\nu}(x, y)^*\,.
\end{aligned}$$

It follows that:

$$\begin{aligned}
\rho_{\mathrm{S}}^{(\mathrm{f})}(x^0, y^0; p) &= \rho_{\mathrm{S}}^{(\mathrm{f})}(x^0, y^0; p)^*\,,\\
\widetilde{\rho}_{\mathrm{V}}^{(\mathrm{f})0}(x^0, y^0; p) &= -\widetilde{\rho}_{\mathrm{V}}^{(\mathrm{f})0}(x^0, y^0; p)^*\,,\\
\rho_{\mathrm{V}}^{(\mathrm{f})}(x^0, y^0; p) &= \rho_{\mathrm{V}}^{(\mathrm{f})}(x^0, y^0; p)^*\,,\\
\rho_{\mathrm{T}}^{(\mathrm{f})}(x^0, y^0; p) &= \rho_{\mathrm{T}}^{(\mathrm{f})}(x^0, y^0; p)^*\,,
\end{aligned}$$

i. e. the temporal vector component is imaginary while all other components are real.

### Statistical Function

One has:

$$\begin{aligned}
F_{\mathrm{S}}^{(\mathrm{f})}(x, y) &= F_{\mathrm{S}}^{(\mathrm{f})}(x, y)^*\,,\\
F_{\mathrm{V}}^{(\mathrm{f})\mu}(x, y) &= -F_{\mathrm{V}}^{(\mathrm{f})\mu}(x, y)^*\,,\\
F_{\mathrm{T}}^{(\mathrm{f})\mu\nu}(x, y) &= -F_{\mathrm{T}}^{(\mathrm{f})\mu\nu}(x, y)^*\,.
\end{aligned}$$

It follows that:

$$\begin{aligned}
F_{\mathrm{S}}^{(\mathrm{f})}(x^0, y^0; p) &= F_{\mathrm{S}}^{(\mathrm{f})}(x^0, y^0; p)^*\,,\\
\widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(x^0, y^0; p) &= -\widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(x^0, y^0; p)^*\,,\\
F_{\mathrm{V}}^{(\mathrm{f})}(x^0, y^0; p) &= F_{\mathrm{V}}^{(\mathrm{f})}(x^0, y^0; p)^*\,,\\
F_{\mathrm{T}}^{(\mathrm{f})}(x^0, y^0; p) &= F_{\mathrm{T}}^{(\mathrm{f})}(x^0, y^0; p)^*\,,
\end{aligned}$$

i. e. the temporal vector component is imaginary while all other components are real.

# Appendix E

# Details of the Numerical Implementation

In this appendix, we present the details of the numerical implementation.

## E.1 Discretization

Here we describe the discretization of the temporal and spatial grids.

### E.1.1 Temporal Grid

#### Real Time Grid: Standard Representation

In the standard representation, time is discretized according to

$$t_i = i\,\Delta t = i\,a_t$$

with the (equidistant) temporal grid spacing $\Delta t = a_t$ and the integer index $i \in \{0, \ldots, N_t - 1\}$ where $N_t$ is the number of time steps. Correspondingly, $t_0$ is the initial time and $t_{\max} := t_{N_t-1} = (N_t - 1)\,a_t$ is the time span.

### E.1.2 Spatial Grid

#### Position Grid

Since we only consider spatially isotropic situations, we employ a spherical grid which can be described by a single coordinate, the radial distance, which we discretize as

$$r_n = \left(n + \frac{1}{2}\right)\Delta r = \left(n + \frac{1}{2}\right) a_s$$

with the (equidistant) spatial grid spacing $\Delta r = a_s$ which corresponds to the distance between two adjacent grid points and the integer index $n \in \{0, \ldots, N_s - 1\}$ where $N_s - 1$

labels the largest distance $r_{\max} := r_{N_s-1} = (N_s - 1/2)\, a_s$, while the smallest distance is given by $r_{\min} := r_0 = a_s/2$ (note that it does not vanish — spherical coordinates are not defined for a vanishing radial distance).

The continuum limit corresponds to the limit $a_s \to 0$, $N_s \to \infty$ for fixed $a_s N_s$. In the continuum limit, we have $r_{\max} = a_s N_s$ and $r_{\min} = 0$.

**Momentum Grid**

Due to spatial homogeneity, we usually work in spatial Fourier space, so the corresponding discretization of the momenta is given by:

$$p_n = (n+1)\Delta p = \frac{\pi}{a_s\, N_s}\,(n+1)$$

with the (again equidistant) momentum grid spacing $\Delta p = \pi/(a_s\, N_s)$ where now $N_s - 1$ labels the largest momentum $\Lambda_{\text{UV}} := p_{\max} := p_{N_s-1} = \pi/a_s$ (the UV cutoff), while the smallest momentum is given by $\Lambda_{\text{IR}} := p_{\min} := p_0 = \pi/(a_s\, N_s) = \Lambda_{\text{UV}}/N_s$ (the IR cutoff). Wen can hence write the momenta as

$$p_n = (n+1)\Lambda_{\text{IR}} = \frac{n+1}{N_s}\,\Lambda_{\text{UV}}\,.$$

It follows from the discretization that there is a finite IR cutoff.

### E.1.3 Functions

After discretization, functions become tensors (whose rank is given by the number of arguments):

$$f(t,t';|\boldsymbol{p}|) \quad \to \quad f(t_i,t_j;p_n) = f_{i,j;n}$$

with $n \in \{0,\ldots,N_s-1\}$ and $j \in \{0,\ldots,N_t-1\}$.

### E.1.4 Derivatives

After discretization, derivatives become difference quotients. We define the forward difference quotient as

$$\Delta_a^{>} f_n = \Delta_{\Delta r}^{>} f(r_n) = \frac{f(r_n+\Delta r) - f(r_n)}{\Delta r} = \frac{f_{n+1} - f_n}{a}\,,$$

and the backward difference quotient correspondingly as

$$\Delta_a^{<} f_n = \frac{f_n - f_{n-1}}{a}\,.$$

Forming forward and backward difference quotients, we obtain a discretization of the second derivative:

$$\Delta_a^{>}\Delta_a^{<} f_n = \frac{f_{n+1} - 2f_n + f_{n-1}}{a^2}\,.$$

For a function of two arguments, we obtain:

$$\Delta^{>}_{(1)a}\Delta^{>}_{(2)a} f_{mn} = \frac{f_{(m+1)(n+1)} - f_{(n+1)m} - f_{n(m+1)} + f_{mn}}{a^2}\,.$$

### E.1.5 Integrals

We will only consider isotropic integrands. After discretization, integrals become sums:

$$\int_0^\infty \quad \to \quad \sum_{n=0}^{N_s-1}\,.$$

#### Position Space

We have for the integration measure:

$$\mathrm{d}^3x = 4\pi\, r^2\, \mathrm{d}r \quad \to \quad 4\pi\, r_n^2 \Delta r = 4\pi \left[a_s\left(n+\frac{1}{2}\right)\right]^2 a_s = 4\pi\, a_s^3\left(n+\frac{1}{2}\right)^2.$$

It follows for a given isotropic function $f$:

$$\int \mathrm{d}^3x\, f(|\boldsymbol{x}|) = 4\pi \int_0^\infty \mathrm{d}r\, r^2 f(r) \quad \to \quad 4\pi\, a_s^3 \sum_{n=0}^{N_s-1}\left(n+\frac{1}{2}\right)^2 f_n\,.$$

The spatial volume in the continuum limit is then given by:

$$V = 4\pi \int_0^{r_{\max}} \mathrm{d}r\, r^2 = \frac{4\pi}{3}\, r_{\max}^3\,.$$

In the discretized case, we obtain:

$$V = 4\pi a_s^3 \sum_{n=0}^{N_s-1}\left(n+\frac{1}{2}\right)^2 = \frac{4\pi}{3}\,(a_s N_s)^3 - \frac{\pi a_s^3 N_s}{3}\,.$$

The relative error is hence given by:

$$\frac{\pi a_s^3 N_s/3}{4\pi a_s^3 N_s^3/3} = \frac{1}{4N_s^2}\,.$$

#### Momentum Space

We have for the integration measure:

$$\begin{aligned}
&\frac{\mathrm{d}^3p}{(2\pi)^3} = \frac{4\pi}{(2\pi)^3}\, p^2\, \mathrm{d}p = \frac{1}{2\pi^2}\, p^2\, \mathrm{d}p \\
\to\quad &\frac{1}{2\pi^2}\, p_n^2\, \Delta p = \frac{1}{2\pi^2}\left[\frac{\pi}{a_s N_s}\,(n+1)\right]^2 \frac{\pi}{a_s N_s} = \frac{\pi}{2(a_s N_s)^3}\,(n+1)^2\,.
\end{aligned}$$

It follows for a given isotropic function $f$:

$$\begin{aligned}
&\int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, f(|\boldsymbol{p}|) = \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}p\, p^2 f(p) \\
\to\quad &\frac{\pi}{2(a_s N_s)^3}\sum_{n=0}^{N_s-1}(n+1)^2 f_n = \frac{1}{2\pi^2}\left(\frac{\pi}{a_s N_s}\right)^3 \sum_{n=0}^{N_s-1}(n+1)^2 f_n\,.
\end{aligned}$$

### E.1.6 Fourier Transformation

After discretization, Fourier transformations become finite-dimensional linear maps.

**Inverse Fourier Transformation (Momentum Space to Position Space)** For a given isotropic function $f$ in momentum space, we have in spherical coordinates $(p, \theta, \varphi)$ with $r = |\boldsymbol{x}|$ and $p = |\boldsymbol{p}|$:

$$\begin{aligned}
\widetilde{f}(r) &= \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\, f(|\boldsymbol{p}|)\, \mathrm{e}^{\mathrm{i}\boldsymbol{p}\cdot\boldsymbol{x}} \\
&= \frac{1}{(2\pi)^3}\int_0^\infty \mathrm{d}\,p\,p^2 \int_0^\pi \mathrm{d}\theta\, \sin(\theta) \int_0^{2\pi} \mathrm{d}\varphi\, f(p)\, \mathrm{e}^{\mathrm{i}\,p\,r\cos(\theta)} \\
&= \frac{1}{(2\pi)^2}\int_0^\infty \mathrm{d}p\,p^2 \int_{-1}^1 \mathrm{d}x\, f(p)\, \mathrm{e}^{\mathrm{i}\,p\,r\,x} \\
&= \frac{1}{4\pi^2\,\mathrm{i}\,r}\int_0^\infty \mathrm{d}p\,p\, f(p)\left(\mathrm{e}^{\mathrm{i}\,p\,r} - \mathrm{e}^{-\mathrm{i}\,p\,r}\right) \\
&= \frac{1}{2\pi^2\,r}\int_0^\infty \mathrm{d}p\,p\,\sin(p\,r) f(p) \\
&= \int_0^\infty \mathrm{d}p\, A(r,p) f(p)
\end{aligned}$$

with the kernel

$$A(r,p) = \frac{p\sin(p\,r)}{2\pi^2 r}\,.$$

Note that, using $\lim_{r\to 0}\sin(p\,r)/r = p$, the limit $r \to 0$ is well-defined:[1]

$$\widetilde{f}(0) = \frac{1}{2\pi^2}\int_0^\infty \mathrm{d}p\, p^2 f(p)\,.$$

[1]Further note that, while in the continuum limit the integral over all momenta is identical to the zero mode of the inverse Fourier transformation, this is not the case for discretization on a grid. The reason is that with our conventions, there is no exact spatial zero mode.

After discretization, functions become $N$-dimensional vectors, and since the Fourier transformation is linear, it can be represented by a matrix:

$$\begin{aligned}
\widetilde{f}_m &= \widetilde{f}(r_m) \\
&= \frac{1}{2\pi^2\, r_m} \sum_{n=0}^{N_s-1} \Delta p\, p_n \sin(p_n\, r_m) f(p_n) \\
&= \frac{1}{2\, a_s^3\, N_s^2 \left(m+\frac{1}{2}\right)} \sum_{n=0}^{N_s-1} \sin\!\left(\frac{\pi}{N_s}\left(m+\frac{1}{2}\right)(n+1)\right)(n+1)\, f_n \\
&= \sum_{n=0}^{N_s-1} A_{mn} f_n\,,
\end{aligned}$$

i. e. $\widetilde{f} = Af$ with the $N_s \times N_s$ matrix $A$ with components

$$A_{mn} = \frac{1}{2\, a_s^3\, N_s^2 \left(m+\frac{1}{2}\right)} \sin\!\left(\frac{\pi}{N_s}\left(m+\frac{1}{2}\right)(n+1)\right)(n+1) = \alpha \left(m+\frac{1}{2}\right)^{-1} A_{mn}^{\text{FFTW}}(n+1)$$

with the (constant, but grid parameter dependent) prefactor

$$\alpha = \frac{1}{4\, a_s^3\, N_s^2}$$

which is determined by the geometry of the grid and the matrix $A^{\text{FFTW}}$ with components

$$A_{mn}^{\text{FFTW}} = 2 \sin\!\left(\frac{\pi}{N}\left(m+\frac{1}{2}\right)(n+1)\right)\,.$$

It is the matrix $A^{\text{FFTW}}$ which is employed by the FFTW library [FFT]. We can hence write

$$\widetilde{f}_m = \alpha \left(m+\frac{1}{2}\right)^{-1} \sum_{n=0}^{N_s-1} A_{mn}^{\text{FFTW}}(n+1) f_n\,.$$

In order to see how to implement this, one has to read the right-hand side of the previous equation from right to left. The algorithm to compute the Fourier transform of the vector $(f_n)$ is therefore as follows:

1. Multiply each of the components $f_n$ of the array $f$ by $(n+1)$.
2. Apply the FFTW Fast Fourier Transformation to the resulting array.
3. Multiply each of the components of the resulting array by the factor $\alpha/(m+1/2)$.

**Fourier Transformation (Position Space to Momentum Space)**

With a similar calculation as for the Fourier transformation, one obtains for the inverse Fourier transformation:

$$f(p) = \frac{4\pi}{p} \int_0^\infty \mathrm{d}r\, r \sin(p\, r)\, \widetilde{f}(r) = \int_0^\infty \mathrm{d}r\, B(p,r) \widetilde{f}(r)$$

with the kernel

$$B(p,r) = \frac{4\pi r \sin(p\,r)}{p}\,.$$

After discretization, one obtains:

$$\begin{aligned} f_m &= f(p_m) \\ &= \frac{4\pi}{p_m}\sum_{n=0}^{N_s-1} \Delta r\, r_n \sin(r_n\, p_m) f(r_n) \\ &= \frac{4a_s^3 N_s}{m+1}\sum_{n=0}^{N_s-1} \sin\!\left(\frac{\pi}{N_s}\left(n+\frac{1}{2}\right)(m+1)\right)\left(n+\frac{1}{2}\right)\tilde{f}_n \\ &= \sum_{n=0}^{N_s-1} B_{mn}\tilde{f}_n \end{aligned}$$

i.e. $f = B\tilde{f}$ with the $N_s \times N_s$ matrix $B$ with components

$$B_{mn} = \frac{4a^3 N_s}{m+1}\sin\!\left(\frac{\pi}{N_s}\left(n+\frac{1}{2}\right)(m+1)\right)\left(n+\frac{1}{2}\right) = \beta\,(m+1)^{-1}\,B^{\text{FFTW}}_{mn}\left(n+\frac{1}{2}\right)$$

with the (constant) prefactor

$$\beta = 2a_s^3 N_s$$

which is determined by the geometry of the grid and the matrix $B^{\text{FFTW}}$ with components

$$B^{\text{FFTW}}_{mn} = 2\sin\!\left(\frac{\pi}{N_s}\left(n+\frac{1}{2}\right)(m+1)\right)\,.$$

It is the matrix $B^{\text{FFTW}}$ which is employed by the FFTW library [FFT]. We can hence write

$$f_m = \beta\,(m+1)^{-1}\sum_{n=0}^{N_s-1} B^{\text{FFTW}}_{mn}\left(n+\frac{1}{2}\right)\tilde{f}_n\,.$$

In order to see how to implement this, one has to read the right-hand side of the previous equation from right to left. The algorithm to compute the Fourier transform of the vector $(\tilde{f}_n)$ is therefore as follows:

1. Multiply each of the components $\tilde{f}_n$ of the array $\tilde{f}$ by $(n+1/2)$.
2. Apply the FFTW Fast Fourier Transformation to the resulting array.
3. Multiply each of the components of the resulting array by the factor $\beta/(m+1)$.

## E.2 Numerical Methods for Solving Differential Equations

The EoMs for the photons and fermions are partial differential equations, second-order in the case of the photons and first-order in the case of the fermions.

There are many algorithms available for numerically solving partial differential equations. The simplest one is the *Euler method*. Unfortunately, however, it turns out that it is not suitable for solving the EoMs, neither for the photons nor for the fermions.

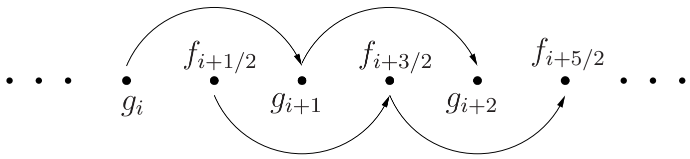



Figure E.1: Sketch of how updating works in the leapfrog algorithm: The two quantities to be evolved "leapfrog" over each other.

### E.2.1 The Leapfrog Method

Since the fermion EoMs are first-order differential equations (i. e. the highest derivative is the first derivative), the Euler algorithm is not stable, and one has to resort to some other method to solve them.

One such method is the leapfrog algorithm. It is well-suited for coupled differential equations of the form[2]

$$\begin{aligned} \dot{f}(f) &= F(g(t))\,, \\ \dot{g}(f) &= G(f(t))\,. \end{aligned}$$

The trick is to discretize $f$ and $g$ on different grid sites:

$$\begin{aligned} g_i &= g_{i-1} + \Delta t\, G(f_{i-1/2})\,, \\ f_{i+1/2} &= f_{i-1/2} + \Delta t\, F(g_i) \end{aligned}$$

(where $\Delta t$ is the time step width). To each quantity, we assign a "leapfrog parity". We call $g$ "leapfrog-even" (or just "even") and $f$ "leapfrog-odd" (or just "odd").[3] The reason that $G$ may not depend on $g$ and $F$ may not depend on $f$ is that $g$ is not defined on odd sites (since it is even) and $f$ is not defined on even sites (since it is odd).[4]

Note that the *free* fermion EoMs have exactly this form. This is easiest to see for massless fermions, i. e. $m^{(f)} = 0$. Then the scalar and tensor components vanish identically,

[2] For the special case $\dot{g} = f$, one obtains $\dot{g} = G(\dot{g})$, so that $G$ is the identity function, and $\ddot{g} = F(g)$. This allows e. g. to solve Newton's equation $\ddot{x} = F(x)$ by solving $\dot{v} = F(x)$ and $\dot{x} = v$.

[3] The nomenclature becomes more clear if we multiply the indices by two. Also note that "parity" is a relative notion, i. e. it does not matter which quantity we call "even" and which we call "odd".

[4] Note that the leapfrog algorithm is very well-suited for solving Hamilton's equations

$$\dot{q} = \frac{\partial H(q,p)}{\partial p}\,, \quad \dot{p} = -\frac{\partial H(q,p)}{\partial q}$$

for the case of a separable Hamiltonian $H(q,p) = T(p) + V(q)$, so that

$$\dot{q} = \frac{\mathrm{d}T(p)}{\mathrm{d}p}\,, \quad \dot{p} = -\frac{\mathrm{d}V(q)}{\mathrm{d}q}\,.$$

and the free EoMs read:

$$\frac{\partial}{\partial t}\widetilde{\rho}_{\mathrm{V}}^{(\mathrm{f})0}(t,t';p) = -p\,\rho_{\mathrm{V}}^{(\mathrm{f})}(t,t';p)\,,$$
$$\frac{\partial}{\partial t}\rho_{\mathrm{V}}^{(\mathrm{f})}(t,t';p) = p\,\widetilde{\rho}_{\mathrm{V}}^{(\mathrm{f})0}(t,t';p)\,.$$

We can therefore assign even leapfrog parity to the spatial vector component and odd leapfrog parity to the temporal vector component. Note that, if we discretize $(t,t') \to (i,j)$, we define a quantity to be leapfrog even/odd if $i-j$ is even/odd.

For massive fermions, the components appear as pairs $(\rho_{\mathrm{S}}^{(\mathrm{f})}, \rho_{\mathrm{V}}^{(\mathrm{f})})$ and $(\rho_{\mathrm{T}}^{(\mathrm{f})}, \widetilde{\rho}_{\mathrm{V}}^{(\mathrm{f})0})$ on the right-hand sides of the equations. Therefore, the scalar component has to be even (like the spatial vector component), and the tensor component has to be odd (like the temporal vector component).

Note, however, that the *full* fermion EoMs do *not* have a form which is perfectly suitable for the application of the leapfrog algorithm, since in the memory integrals, quantities with different leapfrog parity appear. One therefore has to make approximations in the memory integrals. One possible approximation is to neglect contributions with the "wrong" leapfrog parity altogether. Since always half of the quantities contributing to a memory integral have the wrong leapfrog parity, one can then account for the neglected contribution by multiplying the memory integral by two. This is a rather rough estimate, but is nevertheless sufficient in many cases.

There is a potential problem, however, when we have products of fermionic quantities with different leapfrog parity which are not defined. This problem can potentially occur in the photon self-energy which consists of products of components of the fermion propagator. Luckily, it turns out that each component of the photon self-energy except the two vector components consists only of products of fermionic quantities with like Lorentz type (and hence implying the same leapfrog parity). The vector components of the photon self-energy, however, are slightly problematic, since in this case *only* products of fermionic quantities with different leapfrog parity enter.

In order to have a better approximation for the memory integrals and to get nonvanishing memory integrals for the vector components in the first place, it is therefore desirable to be able to approximate quantities of a given leapfrog parity for times at which they are actually not defined. We will do this by expanding quantities around times for which they are not defined. For the free EoMs, this can be done exactly, i. e. without introducing any error at all. For the full equations, this is not possible, however.

We start by noting that from (3.19) it follows that each Lorentz component satisfies a Klein-Gordon type equation:

$$\left[\frac{\partial^2}{\partial t^2} + \left(p^2 + m^{(\mathrm{f})2}\right)\right] F_{\Lambda}^{(\mathrm{f})}(t,t';p) = \widetilde{I}_{(\mathrm{F})\Lambda}^{(\mathrm{f})}(t,t';p)\,,$$

where $\widetilde{I}_{(\mathrm{F})\Lambda}^{(\mathrm{f})}(t,t';p)$ is the corresponding memory integral (note that it is *not* the memory integral of the original, first-order EoM of the respective Lorentz component).

We then have:

$$\frac{\partial^{2n}}{\partial t^{2n}} F_{\Lambda}^{(\mathrm{f})}(t,t';p) = \Big[-\Big(p^2 + m^{(\mathrm{f})2}\Big)\Big]^n F_{\Lambda}^{(\mathrm{f})}(t,t';p) + J_{(\mathrm{F})\,\Lambda}^{(n)}(t,t';p)\,,$$

where $J_{(\mathrm{F})\,\Lambda}^{(n)}(t,t';p)$ contains the memory integral (and terms derived from it, like time derivatives of it) and is therefore suppressed by the coupling (its specific form is not important here), so that

$$\begin{aligned}
&F_{\Lambda}^{(\mathrm{f})}(t-\Delta t,t';p)\\
&\quad= \sum_{n=0}^{\infty} \frac{(-\Delta t)^n}{n!} \frac{\partial^n}{\partial t^n} F_{\Lambda}^{(\mathrm{f})}(t,t';p)\\
&\quad= \sum_{n=0}^{\infty} \left[\frac{(-\Delta t)^{2n}}{(2n)!} \frac{\partial^{2n}}{\partial t^{2n}} + \frac{(-\Delta t)^{2n+1}}{(2n+1)!} \frac{\partial^{2n+1}}{\partial t^{2n+1}}\right] F_{\Lambda}^{(\mathrm{f})}(t,t';p)\\
&\quad= \sum_{n=0}^{\infty} \left[\frac{(\Delta t)^{2n}}{(2n)!} - \frac{(\Delta t)^{2n+1}}{(2n+1)!}\frac{\partial}{\partial t}\right] \left\{\Big[-\Big(p^2 + m^{(\mathrm{f})2}\Big)\Big]^2 F_{\Lambda}^{(\mathrm{f})}(t,t';p) + J_{(\mathrm{F})\,\Lambda}^{(n)}(t,t';p)\right\}\\
&\quad= \sum_{n=0}^{\infty} \Bigg[\frac{(-1)^n\Big(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\Big)^{2n}}{(2n)!}\\
&\qquad\qquad - \frac{(-1)^n\Big(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\Big)^{2n+1}}{(2n+1)!} \frac{1}{\sqrt{p^2+m^{(\mathrm{f})2}}}\frac{\partial}{\partial t}\Bigg] F_{\Lambda}^{(\mathrm{f})}(t,t';p)\\
&\qquad + J_{(\mathrm{F})\Lambda}(t,t';p)\\
&\quad= \left[\cos\Big(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\Big) - \frac{\sin\Big(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\Big)}{\sqrt{p^2+m^{(\mathrm{f})2}}}\frac{\partial}{\partial t}\right] F_{\Lambda}^{(\mathrm{f})}(t,t';p) + J_{(\mathrm{F})\Lambda}(t,t';p)\,.
\end{aligned}$$

From now on, we will neglect $J_{(\mathrm{F})\Lambda}(t,t';p)$, which in a perturbative calculation would correspond to an error of $\mathcal{O}(e^2)$. Then:

$$\begin{aligned}
&F_{\Lambda}^{(\mathrm{f})\,\mathrm{app}}(t,t';p)\\
&\quad= \frac{1}{\cos\Big(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\Big)} \left[F_{\Lambda}^{(\mathrm{f})}(t-\Delta t,t';p) + \frac{\sin\Big(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\Big)}{\sqrt{p^2+m^{(\mathrm{f})2}}} \frac{\partial F_{\Lambda}^{(\mathrm{f})}(t,t';p)}{\partial t}\right]\\
&\quad= \sec\Big(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\Big) F_{\Lambda}^{(\mathrm{f})}(t-\Delta t,t';p) + \frac{\tan\Big(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\Big)}{\sqrt{p^2+m^{(\mathrm{f})2}}} \frac{\partial F_{\Lambda}^{(\mathrm{f})}(t,t';p)}{\partial t}\,.
\end{aligned}$$

The important point here is that the left-hand side and the right-hand side have opposite leapfrog parity. We obtain:

$$\begin{aligned}
&F_{\mathrm{S}}^{(\mathrm{f})\,\mathrm{app}}(t,t';p)\\
&\quad= \frac{1}{\cos\Big(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\Big)} \Big\{F_{\mathrm{S}}^{(\mathrm{f})}(t-\Delta t,t';p)
\end{aligned}$$

$$
\left. + \frac{\sin\left(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\right)}{\sqrt{p^2+m^{(\mathrm{f})2}}} \left[ m^{(\mathrm{f})} \widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(t,t';p) + pF_{\mathrm{T}}^{(\mathrm{f})}(t,t';p) \right] \right\}
$$

$$
\begin{aligned}
&= \sec\left(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\right) F_{\mathrm{S}}^{(\mathrm{f})}(t-\Delta t,t';p) \\
&\quad + \frac{\tan\left(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\right)}{\sqrt{p^2+m^{(\mathrm{f})2}}} \left[ m^{(\mathrm{f})} \widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(t,t';p) + pF_{\mathrm{T}}^{(\mathrm{f})}(t,t';p) \right],
\end{aligned}
$$

$$
\begin{aligned}
&\widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0\mathrm{app}}(t,t';p) \\
&= \frac{1}{\cos\left(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\right)} \left\{ \widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(t-\Delta t,t';p) \right. \\
&\qquad \left. - \frac{\sin\left(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\right)}{\sqrt{p^2+m^{(\mathrm{f})2}}} \left[ m^{(\mathrm{f})} F_{\mathrm{S}}^{(\mathrm{f})}(t,t';p) + pF_{\mathrm{V}}^{(\mathrm{f})}(t,t';p) \right] \right\},
\end{aligned}
$$

$$
\begin{aligned}
&F_{\mathrm{V}}^{(\mathrm{f})\mathrm{app}}(t,t';p) \\
&= \frac{1}{\cos\left(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\right)} \left\{ F_{\mathrm{V}}^{(\mathrm{f})}(t-\Delta t,t';p) \right. \\
&\qquad \left. + \frac{\sin\left(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\right)}{\sqrt{p^2+m^{(\mathrm{f})2}}} \left[ p\widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(t,t';p) - m^{(\mathrm{f})} F_{\mathrm{T}}^{(\mathrm{f})}(t,t';p) \right] \right\},
\end{aligned}
$$

$$
\begin{aligned}
&F_{\mathrm{T}}^{(\mathrm{f})\mathrm{app}}(t,t';p) \\
&= \frac{1}{\cos\left(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\right)} \left\{ F_{\mathrm{T}}^{(\mathrm{f})}(t-\Delta t,t';p) \right. \\
&\qquad \left. + \frac{\sin\left(\sqrt{p^2+m^{(\mathrm{f})2}}\,\Delta t\right)}{\sqrt{p^2+m^{(\mathrm{f})2}}} \left[ -pF_{\mathrm{S}}^{(\mathrm{f})}(t,t';p) + m^{(\mathrm{f})} F_{\mathrm{V}}^{(\mathrm{f})}(t,t';p) \right] \right\}.
\end{aligned}
$$

Note that for $\Delta t = 0$, we obviously have $F_{\Lambda}^{(\mathrm{f})\mathrm{app}}(t,t';p) = F_{\Lambda}^{(\mathrm{f})}(t,t';p)$. Further, for free quantities, we have $F_{\Lambda\,0}^{(\mathrm{f})\,\mathrm{app}}(t,t';p) = F_{\Lambda\,0}^{(\mathrm{f})}(t,t';p)$ for *each* value of $\Delta t$ (i. e. not just for *small* values of $\Delta t$), which can easily be checked by inserting the free expressions and employing addition theorems.

If we were to calculate the photon self-energy perturbatively so that only free fermion propagators enter which are known analytically, an even better solution would be to symmetrize the first solution by evaluating the sum of the undefined fermion propagator one time step earlier and one time step later, divided by two. However, this is of course impossible in a self-consistent calculation where the fermion propagator for the next time step is not known (in fact, we need the self-energy we are to calculate in order to determine the propagator at the next time step).

Note that we obtain the same error if we symmetrize with respect to the to time arguments according to

$$
\widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(t,t') \to \quad \frac{1}{2}\left[ \widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(t-\Delta t,t') + \widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(t,t'-\Delta t) \right]
$$

$$= \frac{1}{2}\left[\widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(t,t') - \Delta t\,\frac{\partial}{\partial t}\widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(t,t') + \widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(t,t') - \Delta t\,\frac{\partial}{\partial t'}\widetilde{F}_{\mathrm{V}}^{(\mathrm{f})0}(t,t') + \mathcal{O}\Big((\Delta t)^2\Big)\right].$$

### E.2.2 The Runge–Kutta Method

The photon EoMs (3.48) and (3.49) are second-order partial differential equations which contain first derivatives as well. It turns out that the fourth-order Runge–Kutta method is best suited for the equations. If we ignore for a moment the fact that the photon EoMs couple different components to each other, the equation for a single component is of the form

$$\ddot{f}(t) = F(t, f(t), \dot{f}(t))\,.$$

The first step is to convert the single second-order equation to two first-order equations

$$\begin{aligned}\dot{g}(t) &= F(t, f(t), g(t))\,,\\ \dot{f}(t) &= G(t, f(t), g(t)) = g(t)\,.\end{aligned}$$

Note that this conversion is always possible and is just a special case (in the sense that $G$ is not arbitrary, but just returns its last argument) of two coupled first-order differential equations. The Runge–Kutta algorithm then has to be applied to both equations. The algorithm reads:

$$\begin{aligned}g_{i+1} &= g_i + \frac{1}{6}\left(k_F^{(1)} + 2k_F^{(2)} + 2k_F^{(3)} + k_F^{(4)}\right),\\ f_{i+1} &= f_i + \frac{1}{6}\left(k_G^{(1)} + 2k_G^{(2)} + 2k_G^{(3)} + k_G^{(4)}\right),\\ t_{i+1} &= t_i + h\,.\end{aligned}$$

with

$$\begin{aligned}k_F^{(1)} &= h\,F(t_i, f_i, g_i)\,,\\ k_G^{(1)} &= h\,G(t_i, f_i, g_i) = h\,g_i\,,\\ k_F^{(2)} &= h\,F\Big(t_i + h/2, f_i + k_G^{(1)}/2, g_i + k_F^{(1)}/2\Big) = h\,F\Big(t_i + h/2, f_i + h\,g_i/2, g_i + k_F^{(1)}/2\Big)\,,\\ k_G^{(2)} &= h\,G\Big(t_i + h/2, f_i + k_G^{(1)}/2, g_i + k_F^{(1)}/2\Big) = h\left(g_i + k_F^{(1)}/2\right),\\ k_F^{(3)} &= h\,F\Big(t_i + h/2, f_i + k_G^{(2)}/2, g_i + k_F^{(2)}/2\Big)\\ &= h\,F\Big(t_i + h/2, f_i + h\left(g_i + k_F^{(1)}/2\right)/2, g_i + k_F^{(2)}/2\Big)\,,\\ k_G^{(3)} &= h\,G\Big(t_i + h/2, f_i + k_G^{(2)}/2, g_i + k_F^{(2)}/2\Big) = h\left(g_i + k_F^{(2)}/2\right),\\ k_F^{(4)} &= h\,F\Big(t_i + h, f_i + k_G^{(3)}, g_i + k_F^{(3)}\Big)\\ &= h\,F\Big(t_i + h, f_i + h\left(g_i + k_F^{(2)}/2\right), g_i + k_F^{(3)}\Big)\,,\\ k_G^{(4)} &= h\,G\Big(t_i + h, f_i + k_G^{(3)}, g_i + k_F^{(3)}\Big) = h\left(g_i + k_F^{(3)}\right),\end{aligned}$$

where the second lines of each $k_G^{(i)}$ can be used to remove any direct reference to $k_G^{(i)}$ in the computation of the $k_F^{(i)}$, which can be seen in the second lines of each $k_F^{(i)}$.

It is easy to see that this method contains corrections to the Euler method by noting that

$$f_{i+1} = f_i + h\, g_i + \frac{h}{6}\left(k_F^{(1)} + k_F^{(2)} + k_F^{(3)}\right),$$

where the last term on the right-hand side would be missing in the Euler method.

Note that the only explicit time dependence in the photon EoMs resides in the memory integrals (i. e. there is no explicit time dependence in the free photon EoMs, as is to be expected). Further note that for the free EoMs, besides the EoM for the transverse component, there are two sets of equations: One coupling the scalar and the type-1 vector component, and one coupling the type-2 vector and the longitudinal component. The two sets decouple in the free case. The general structure of each of the two sets of equations is given by:

$$\begin{aligned}
\ddot{f}_1(t) &= F_1(t, f_1(t), g_2(t)) = \widetilde{F}_1(f_1(t), g_2(t)) + I_1(t)\,, \\
\ddot{f}_2(t) &= F_2(t, f_2(t), g_1(t)) = \widetilde{F}_2(f_2(t), g_1(t)) + I_2(t)\,.
\end{aligned}$$

One then has to solve the following system of four first-order differential equations:

$$\begin{aligned}
\dot{g}_1(t) &= \widetilde{F}_1(f_1(t), g_2(t)) + I_1(t)\,, \\
\dot{f}_1(t) &= g_1(t)\,, \\
\dot{g}_2(t) &= \widetilde{F}_2(f_2(t), g_1(t)) + I_2(t)\,, \\
\dot{f}_2(t) &= g_2(t)\,.
\end{aligned}$$

Note that $\widetilde{F}_i$ $(i = 1, 2)$ does not depend explicitly on time; only the memory integrals $I_i$ do.

The auxiliary variables read:

$$\begin{aligned}
k_{F_1}^{(1)} &= h\, F_1(t_i, f_{1i}, g_{2i}) = h\left[\widetilde{F}_1(f_{1i}, g_{2i}) + I_1(t_i)\right] = \widetilde{k}_{F_1}^{(1)} + h\, I_1(t_i)\,, \\
k_{G_1}^{(1)} &= h\, g_{1i}\,, \\
k_{F_2}^{(1)} &= h\, F_2(t_i, f_{2i}, g_{1i}) = h\left[\widetilde{F}_2(f_{2i}, g_{1i}) + I_2(t_i)\right] = \widetilde{k}_{F_2}^{(1)} + h\, I_2(t_i)\,, \\
k_{G_2}^{(1)} &= h\, g_{2i}\,, \\
k_{F_1}^{(2)} &= h\, F_1\Big(t_i + h/2, f_{1i} + k_{G_1}^{(1)}/2, g_{2i} + k_{F_2}^{(1)}/2\Big) \\
&= h\left[\widetilde{F}_1\Big(f_{1i} + k_{G_1}^{(1)}/2, g_{2i} + k_{F_2}^{(1)}/2\Big) + I_1(t_i + h/2)\right] \\
&= \widetilde{k}_{F_1}^{(2)} + h\, I_1(t_i + h/2)\,, \\
k_{G_1}^{(2)} &= h\left(g_{1i} + k_{F_1}^{(1)}/2\right), \\
k_{F_2}^{(2)} &= h\, F_2\Big(t_i + h/2, f_{2i} + k_{G_2}^{(1)}/2, g_{1i} + k_{F_1}^{(1)}/2\Big)
\end{aligned}$$

$$
\begin{aligned}
&= h\left[\widetilde{F}_2\Big(f_{2i} + k_{G_2}^{(1)}/2, g_{1i} + k_{F_1}^{(1)}/2\Big) + I_2(t_i + h/2)\right] \\
&= \widetilde{k}_{F_2}^{(2)} + h\, I_2(t_i + h/2)\,, \\
k_{G_2}^{(2)} &= h\left(g_{2i} + k_{F_2}^{(1)}/2\right), \\
k_{F_1}^{(3)} &= h\, F_1\Big(t_i + h/2, f_{1i} + k_{G_1}^{(2)}/2, g_{2i} + k_{F_2}^{(2)}/2\Big) \\
&= h\left[\widetilde{F}_1\Big(f_{1i} + k_{G_1}^{(2)}/2, g_{2i} + k_{F_2}^{(2)}/2\Big) + I_1(t_i + h/2)\right] \\
&= \widetilde{k}_{F_1}^{(3)} + h\, I_1(t_i + h/2)\,, \\
k_{G_1}^{(3)} &= h\left(g_{1i} + k_{F_1}^{(2)}/2\right), \\
k_{F_2}^{(3)} &= h\, F_2\Big(t_i + h/2, f_{2i} + k_{G_2}^{(2)}/2, g_{1i} + k_{F_1}^{(2)}/2\Big) \\
&= h\left[\widetilde{F}_2\Big(f_{2i} + k_{G_2}^{(2)}/2, g_{1i} + k_{F_1}^{(2)}/2\Big) + I_2(t_i + h/2)\right] \\
&= \widetilde{k}_{F_2}^{(3)} + h\, I_2(t_i + h/2)\,, \\
k_{G_2}^{(3)} &= h\left(g_{2i} + k_{F_2}^{(2)}/2\right), \\
k_{F_1}^{(4)} &= h\, F_1\Big(t_i + h, f_{1i} + k_{G_1}^{(3)}, g_{2i} + k_{F_2}^{(3)}\Big) \\
&= h\left[\widetilde{F}_1\Big(f_{1i} + k_{G_1}^{(3)}, g_{2i} + k_{F_2}^{(3)}\Big) + I_1(t_i + h)\right] \\
&= \widetilde{k}_{F_1}^{(4)} + h\, I_1(t_i + h)\,, \\
k_{G_1}^{(4)} &= h\left(g_{1i} + k_{F_1}^{(3)}\right), \\
k_{F_2}^{(4)} &= h\, F_2\Big(t_i + h/2, f_{2i} + k_{G_2}^{(3)}, g_{1i} + k_{F_1}^{(3)}\Big) \\
&= h\left[\widetilde{F}_2\Big(f_{2i} + k_{G_2}^{(3)}, g_{1i} + k_{F_1}^{(3)}\Big) + I_2(t_i + h)\right] \\
&= \widetilde{k}_{F_2}^{(4)} + h\, I_2(t_i + h)\,, \\
k_{G_2}^{(4)} &= h\left(g_{2i} + k_{F_2}^{(3)}\right).
\end{aligned}
$$

Note, however, that we cannot evaluate the memory integrals at $t_i + h/2$ since they depend on the correlation functions and are hence only defined on the grid points. We also cannot evaluate the memory integrals at $t_i + h$. Since it is equal to $t_{i+1}$, it does correspond to a grid point, but would make the equations implicit. We therefore make the approximation that we evaluate the memory integrals only at $t_i$. This leads to some sort of a hybrid approach where we use the RK4 method for the free part of the EOMs and the Euler method for the interacting part.

We then have:

$$
\begin{aligned}
g_{1(i+1)} &= g_{1i} + \frac{1}{6}\left(k_{F_1}^{(1)} + 2k_{F_1}^{(2)} + 2k_{F_1}^{(3)} + k_{F_1}^{(4)}\right) \\
&= g_{1i} + \frac{1}{6}\left(\widetilde{k}_{F_1}^{(1)} + 2\widetilde{k}_{F_1}^{(2)} + 2\widetilde{k}_{F_1}^{(3)} + \widetilde{k}_{F_1}^{(4)}\right) + h\, I_1(t_i)\,,
\end{aligned}
$$

$$
\begin{aligned}
f_{1(i+1)} &= f_{1i} + \frac{1}{6}\left(k_{G_1}^{(1)} + 2k_{G_1}^{(2)} + 2k_{G_1}^{(3)} + k_{G_1}^{(4)}\right), \\
g_{2(i+1)} &= g_{2i} + \frac{1}{6}\left(k_{F_2}^{(1)} + 2k_{F_2}^{(2)} + 2k_{F_2}^{(3)} + k_{F_2}^{(4)}\right) \\
&= g_{2i} + \frac{1}{6}\left(\widetilde{k}_{F_2}^{(1)} + 2\widetilde{k}_{F_2}^{(2)} + 2\widetilde{k}_{F_2}^{(3)} + \widetilde{k}_{F_2}^{(4)}\right) + h\, I_2(t_i)\,, \\
f_{2(i+1)} &= f_{2i} + \frac{1}{6}\left(k_{G_2}^{(1)} + 2k_{G_2}^{(2)} + 2k_{G_2}^{(3)} + k_{G_2}^{(4)}\right).
\end{aligned}
$$

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# Lebenslauf

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## Veröffentlichungen

1. J. Berges, D. Sexty and T. Zöller, *in preparation*

# Eidesstattliche Erklärung

Hiermit erkläre ich eidesstattlich, dass ich die vorliegende Dissertation selbstständig verfasst, keine anderen als die angegebenen Hilfsmittel verwendet und noch keinen Promotionsversuch unternommen habe.

Darmstadt, den 30. Januar 2013,

.................................... gez. Thorsten Zöller

Unterschrift